Structural Testing

Part 2 · Modal Analysis and Simulation
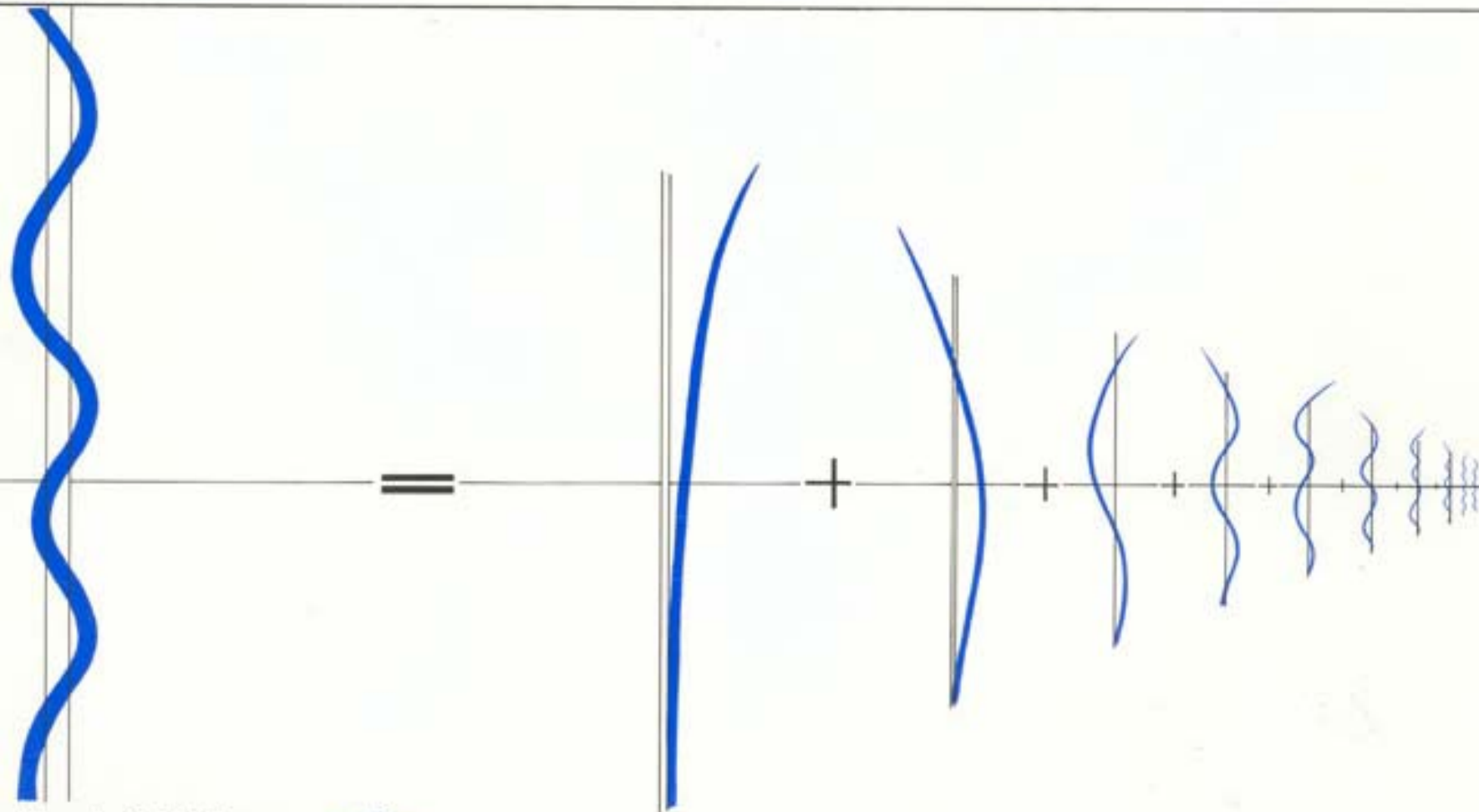
Brüel & Kjær

STRUCTURAL TESTING

Part II: Modal Analysis and Simulation

by

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Preface to Part 2

The study of structural dynamics is essential for understanding and evaluating the performance of any engineering product. Whether we are concerned with printed-circuit boards or suspension bridges, high-speed printer mechanisms or satellite launchers, dynamic response is fundamental to sustained and satisfactory operation.

Modal analysis of the data obtained from structural testing, provides us with a definitive description of the response of a structure, which can be evaluated against design specifications. It also enables us to construct a powerful tool, the modal model, with which we can investigate the effects of structural modifications, or predict how the structure will perform under changed operating conditions.

A simplified definition of modal analysis can be made by comparing it to frequency analysis. In frequency analysis, a complex signal is resolved into a set of simple sine waves with individual frequency and amplitude parameters. In modal analysis a complex deflection pattern (of a vibrating structure) is resolved into a set of simple mode shapes with individual frequency and damping parameters.

A rigorous mathematical approach to the subject is outside the scope of this primer. Where necessary we simply quote the mathematics! definitions required to support our intuitive introductions. For readers who need to confirm the quoted mathematical "truths" we have included a list of relevant literature at the end of this booklet.

To the assumptions already made about the reader (see preface to part 1), we now add a knowledge of the subject matter of part 1 *(Mechanical Mobility Measurements)*.

Experimental Modal Analysis

Introduction

Most structures vibrate. In operation, all machines, vehicles and buildings are subjected to dynamic forces which cause vibrations. Very often the vibrations have to be investigated, either because they cause an immediate problem, or because the structure has to be "cleared" to a "standard" or test specification. Whatever the reason, we need to quantify the structural response in some way, so that its implication on factors such as performance and fatigue can be evaluated.

By using signal-analysis techniques, we can measure vibration on the operating structure and make a frequency analysis. The *frequency spectrum* description of how the vibration level varies with frequency can then be checked against a specification. This type of testing will give results which are only relevant to the measured conditions. The result will be a product of the structural response and the spectrum of an unknown excitation force, it will give little or no information about the characteristics of the structure itself.

An alternative approach is the system-analysis technique in which a dual-channel FFT analyser can be used to measure the ratio of the response to a measured input force. The *frequency response function* (FRF) measurement removes the force spectrum from the data and describes the inherent structural response between the measurement points. From a set of FRF measurements made at defined points on a structure, we can begin to build up a picture of its response. The technique used to do this is modal analysis.

All Structures Exhibit Modal Behaviour

An FRF measurement made on any structure will show its response to be a series of peaks. The individual peaks are often sharp, with identifiable centre-frequencies, indicating that they are resonances, each typical of the response of a single-degree-of-freedom (SDOF) structure. If the broader peaks in the FRF are analyzed with increased frequency resolution, two or more resonances are usually found close together. The implication is that a structure behaves as if it is a set of SDOF substructures. This is the basis of modal analysis, through which the behaviour of a structure can be analyzed by identifying and evaluating all the resonances, or modes, in its response.

Let us begin with a review of how structural response can be represented in different domains. Through this we will be able to see how the modal description relates to descriptions in the spatial, time and frequency domains.

As our example, we will take the response of a bell, which is a lightly damped structure. When the bell is struck, it produces an acoustical response containing a limited number of pure tones. The associated vibration response has exactly the same pattern, and the bell seems to store the energy from the impact and dissipate it by vibrating at particular discrete frequencies.

In the illustration, each column shows the response of the bell represented in different domains:

In the physical domain, the complex geometrical deflection pattern of the bell, can be represented by a set of simpler, independent deflection patterns, or mode shapes.

In the time domain the vibration (or acoustic) response of the bell is shown as a time history, which can be represented by a set of a decaying sinusoids.

In the frequency domain, analysis of the time signal gives us a spectrum containing a series of peaks, shown below as a set of SDOF response spectra.

In the modal domain we see the *response* of the bell as a modal model constructed from a set of SDOF models. Since a mode shape is the pattern of movement for *all the points* on the structure at a modal frequency, a single modal coordinate q can be used to represent the entire movement contribution of each mode.

Looking back from the modal domain, along the rows in the illustration, we see that each SDOF model is associated with a *frequency, a clamping* and a *mode shape.* These are the MODAL PARAMETERS:

- modal frequency

- modal damping

- mode shape

which together form a complete description of the inherent dynamic characteristics of the bell, and are constant whether the bell is ringing or not.

Modal analysis is the process of determining the modal parameters of a structure for all modes in the frequency range of interest. The ultimate goal is to use these parameters to construct a *modal model* of the response.

Two observations worth noting here are that:

- Any forced dynamic deflection of a structure can be represented as a weighted sum of its mode shapes.

- Each mode can be represented by an SDOF model.

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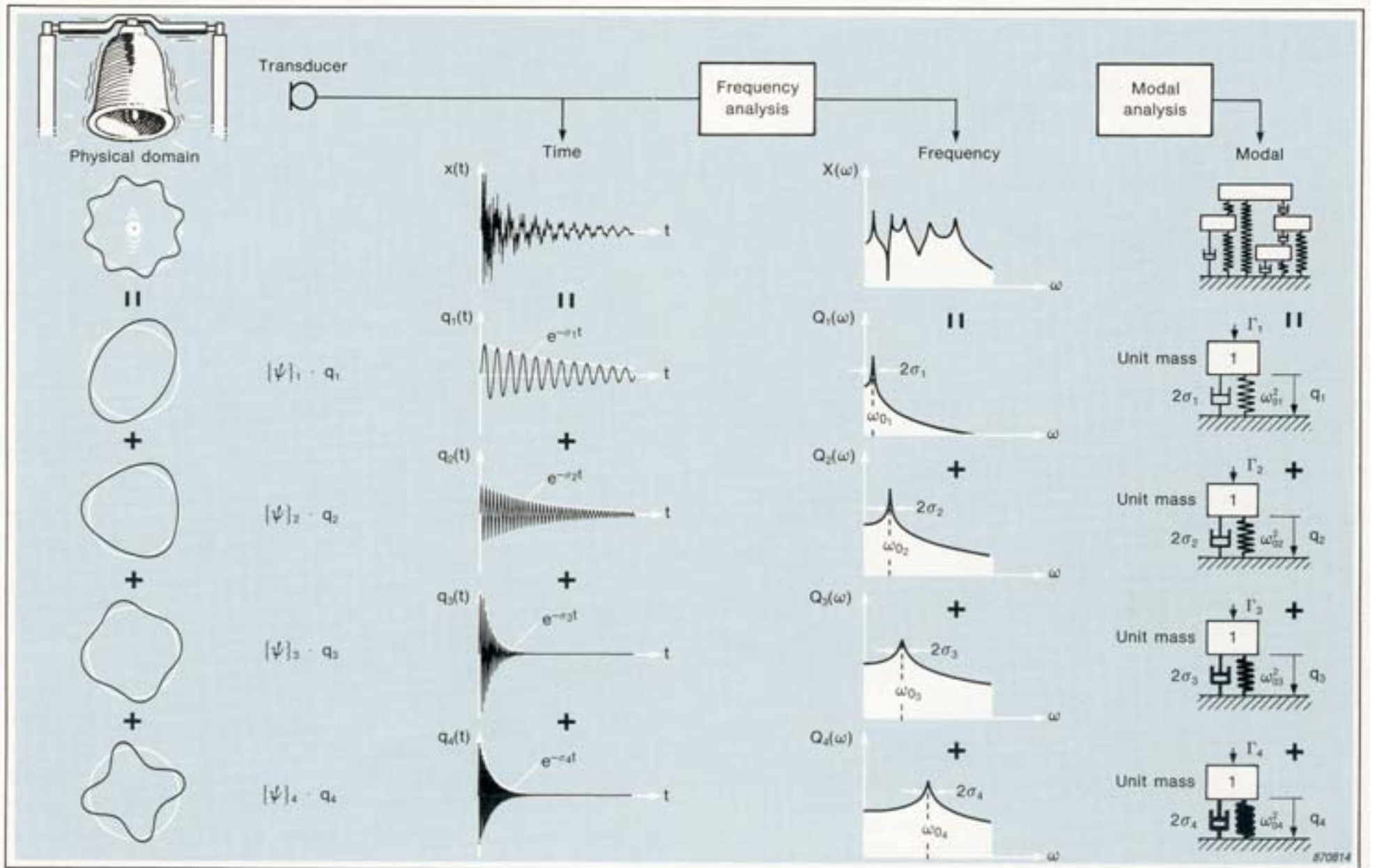
Single-degree-of-freedom (SDOF) Models

As each peak - or mode - in a structural response can be represented by an SDOF model, we will look at some aspects of SDOF dynamics. In particular, we will examine the way in which SDOF structure can be modelled in the physical, time and frequency domains. These models are not intended to represent physical structures, but will serve as instruments for interpreting dynamic behaviour (constrained by a set of assumptions and boundary conditions). They will help us to:

• understand and interpret the behaviour of structures;
• describe the dynamic properties of structures, using a small set of parameters;
• extract the parameters from measured data (curve-fitting).

● **An analytical model** can be constructed in the physical domain. It is an abstract system consisting of a *point mass (m).* supported by a *massless linear spring (k)* and connected to a linear *viscous damper (c).* The mass is constrained so that it can move in only one direction (x) - a *Single-degree-of-freedom.*

● **A mathematical model** in the time domain can be derived by applying Newton's Second Law to the analytical model. By equating the internal forces (inertia, damping and elasticity) with the external (excitation) force, we obtain the model

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

which is a second-order differential equation. A model which is more mathematically manageable can be obtained in the frequency domain.

Models are to be used, but not to be believed. Henri Thiel

Analytical Model

f(t)

m   x(t)

c   k

Newton's Second Law

Mathematical Model
(time domain)

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

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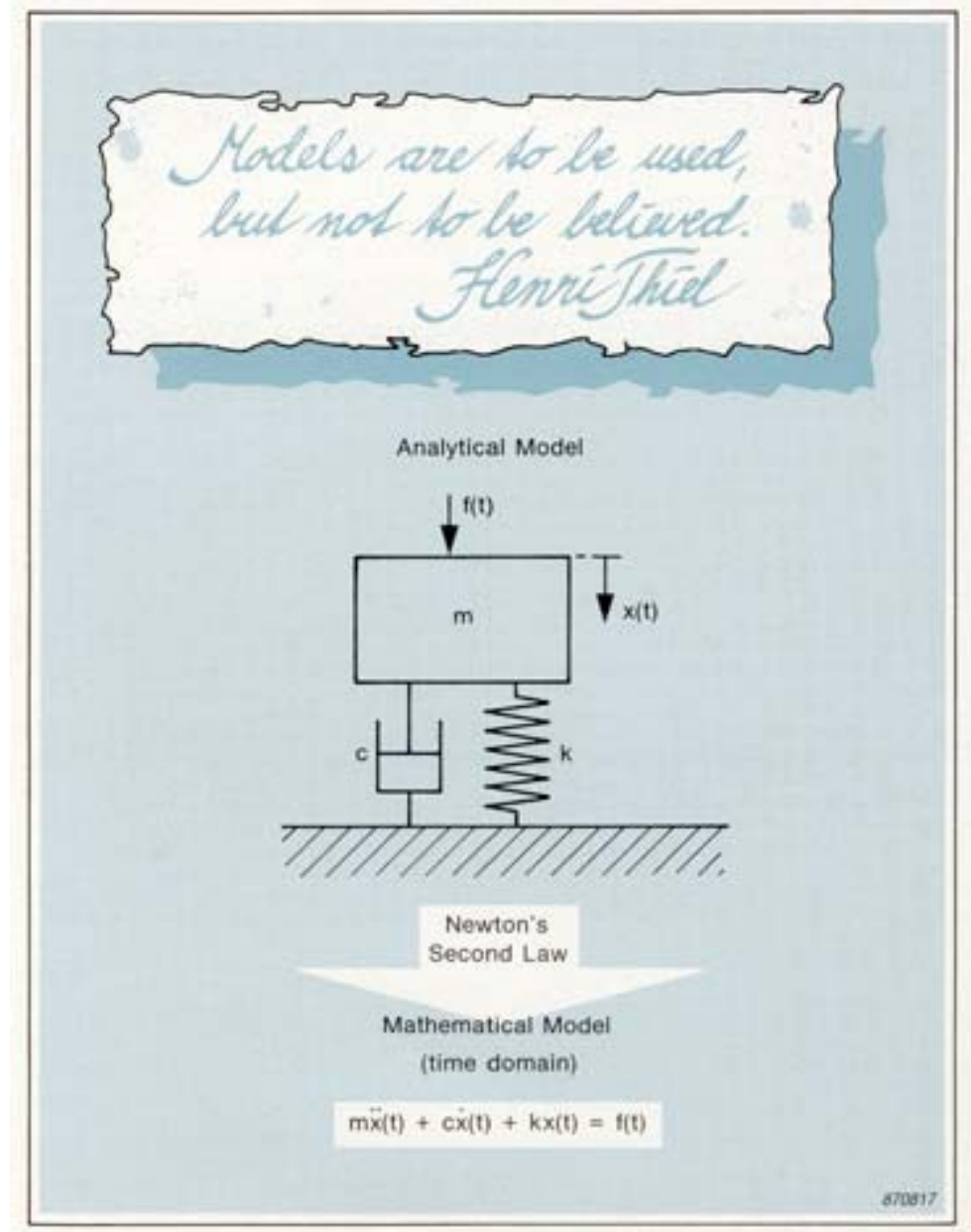
SDOF Models in the Frequency Domain

- **A spatial-parameter model** can be constructed in the frequency domain to describe the frequency response function $H(\omega)$ in terms of mass, spring stiffness and damping coefficient.

Let us look at the behaviour of this model under sinusoidal excitation and see what happens to the magnitude $|H(\omega)|$ and phase $\angle H(\omega)$ as frequency increases.

The static deflection is controlled by spring stiffness alone. At low frequencies the response is dominated by the spring and is in phase with the excitation.

As frequency increases, the inertial force of the mass has an increasing influence. At a particular frequency ($\omega_0 = k/m$ the *undamped natural frequency)* the mass and spring terms cancel each other out, the response is controlled only by the damping term, and compliance becomes high. If the damping term was, in fact, zero the compliance would be infinite. At ω_0, the response lags the excitation by 90°.

At frequencies greater than ω_0, the mass term takes control, the system begins to behave simply as a mass, the compliance decreases and the response lags the excitation by 180°.

- **The FRF (or black-box) model** is non-parametric. It is based on the definition of $H(\omega)$.

$$X(\omega) = H(\omega) \cdot F(\omega)$$

$H(\omega)$ is in terms of compliance (displacement/force). It is the ratio of output/input spectra; and varies as a function of frequency (ω).

This model links the analytical SDOF model to practical measurements.

Spatial parameter model

$$H(\omega) = \frac{1}{-\omega^2 m + j\omega c + k}$$

Spring $\left(\frac{1}{k}\right)$

Damper $\left(\frac{1}{j\omega c}\right)$

Mass $\left(\frac{1}{-\omega^2 m}\right)$

Black box model

$F(\omega)$ $H(\omega)$ $X(\omega)$

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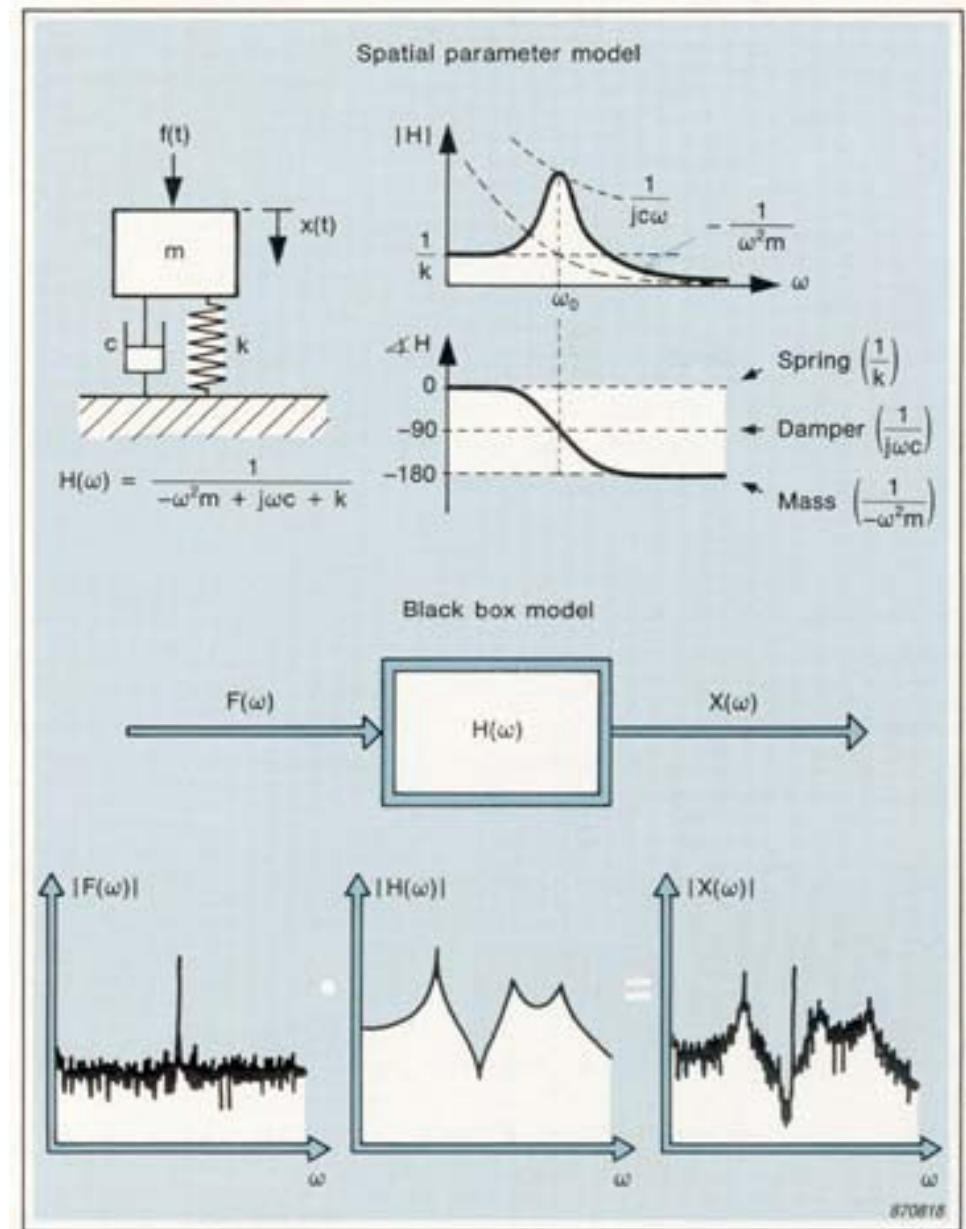
The spatial parameter model is ideal for working with analytical systems. With real structures we will usually have little or no knowledge of the mass, stiffness and damping distributions. The next model is a practical link between theory and measurements.

● **The modal-parameter model** is shown in the illustration. It is constructed using two parameters which can be obtained from FRF measurements.

In the illustration, $H(\omega)$ is defined in terms of the *pole location (p)* and the *residue (R)* - and their complex conjugates *(p** and *R*)*. The pole location and the residue are themselves defined in terms of the spatial parameters.
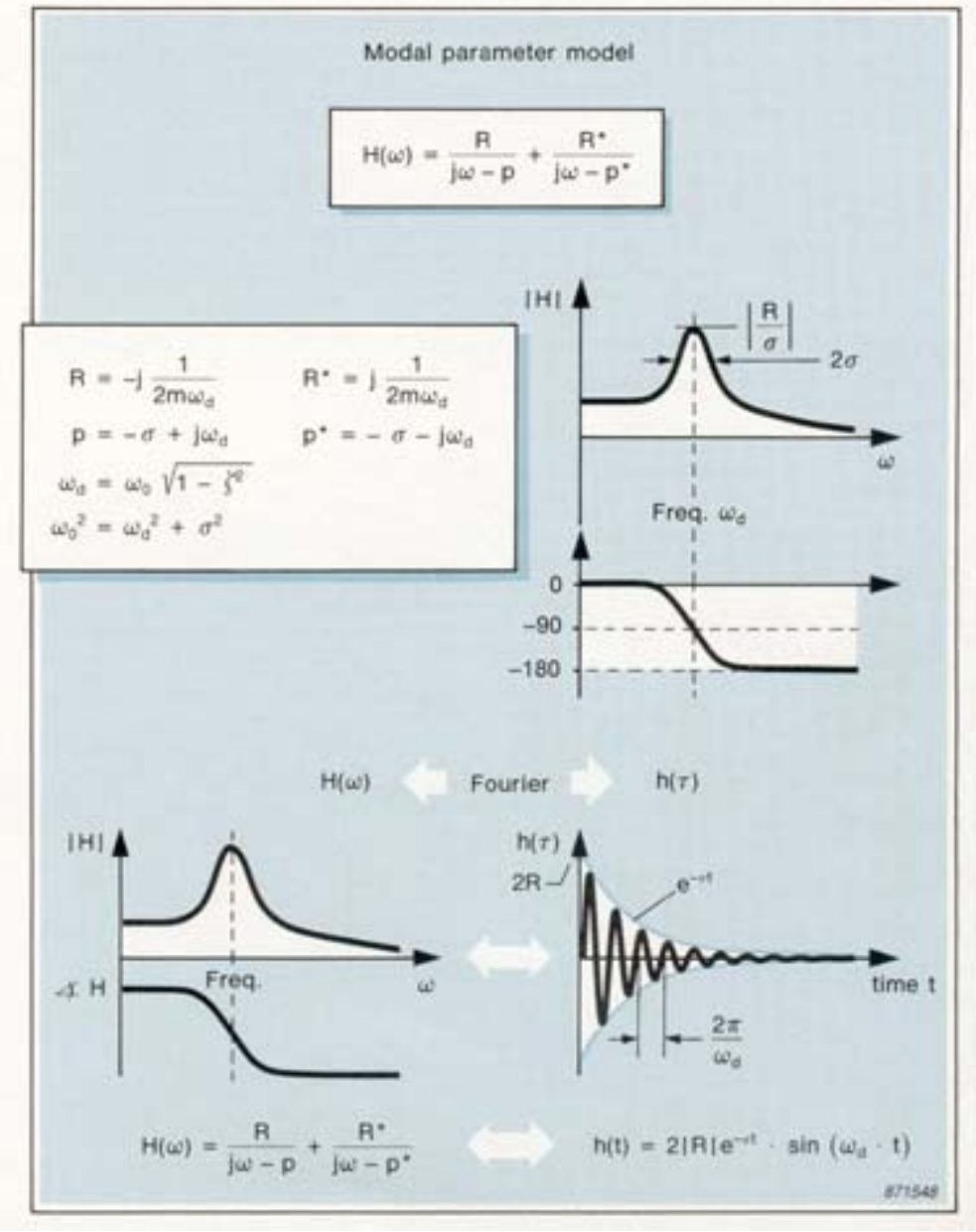
The pole location is a complex number. The numerical value of its real part (σ) is the rate at which a damped oscillation decays. This is shown on the impulse response function in the time domain. In the frequency domain, *a* represents half the -3 dB bandwidth of the FRF peak. The imaginary part of the pole location is the modal frequency - the *damped natural frequency* (ω_d) for a free decaying oscillation.

The residue for an SDOF system is an imaginary number which expresses the strength of the mode.

As indicated in the illustration, both the pole location and the residue can be obtained from measurements made on a displayed FRF. The modal parameter model thus gives the relationship between the analytical models and experimental measurements.

Modal parameter model

$$H(\omega) = \frac{R}{j\omega - p} + \frac{R^*}{j\omega - p^*}$$

$$R = -j\,\frac{1}{2m\omega_d} \qquad R^* = j\,\frac{1}{2m\omega_d}$$

$$p = -\sigma + j\omega_d \qquad p^* = -\sigma - j\omega_d$$

$$\omega_d = \omega_0\sqrt{1 - \zeta^2}$$

$$\omega_0^2 = \omega_d^2 + \sigma^2$$

$H(\omega)$ ◄ Fourier ► $h(\tau)$

$$H(\omega) = \frac{R}{j\omega - p} + \frac{R^*}{j\omega - p^*}$$

$$h(t) = 2|R|e^{-\sigma t}\cdot\sin(\omega_d\cdot t)$$

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A Closer Look at Pole Location and Residue

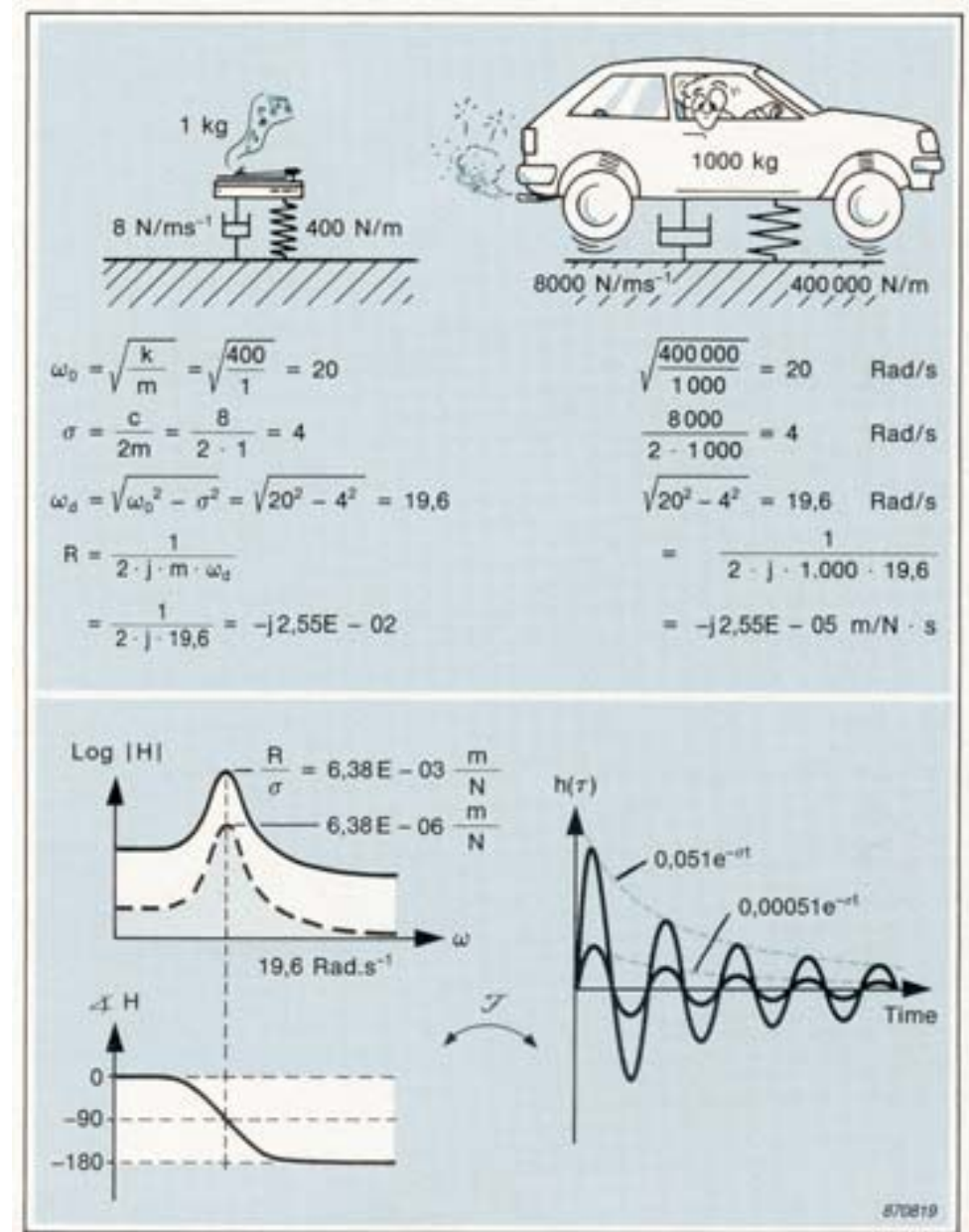As these two parameters are fundamental to modal analysis, let us look at them in more detail.

● **The pole location** contains two of the modal parameters listed on page 4. The real part of the pole location is the rate at which free vibrations die out (related to the modal damping), and the imaginary part is the frequency at which the system oscillates in free decay (modal frequency). This information is held in the form of a centre frequency and a half-bandwidth (at -3 dB) of a resonance. The pole location describes the shape of the magnitude and phase curves of the FRF. It gives us a qualitative measure of the dynamic properties.

● **The residue** is a mathematical concept and has no direct interpretation in physical terms. It carries the absolute scaling of the FRF, and thus the level of the magnitude curve. We will see later that the residue is related to the third modal parameter, the mode shape.

The residue is sometimes called the *pole-strength,* but the magnitude of a mode is not given by the residue alone. It is the ratio of the residue to the decay rate:

$$H(\omega_d) \cong \frac{R}{\sigma}$$

The illustration gives an example of the properties of pole location and residue. In a simplistic SDOF view, a hi-fi turntable unit could have the same stiffness, damping and mass distribution as a car, and hence the same pole location. Their measured FRFs would have the same shape but their response to unit force would be quite different. The difference can be seen in the residues.

1 kg   1000 kg

8 N/ms⁻¹   400 N/m   8000 N/ms⁻¹   400 000 N/m

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{1}} = 20 \qquad \sqrt{\frac{400\,000}{1\,000}} = 20 \quad \text{Rad/s}$$

$$\sigma = \frac{c}{2m} = \frac{8}{2 \cdot 1} = 4 \qquad \frac{8000}{2 \cdot 1000} = 4 \quad \text{Rad/s}$$

$$\omega_d = \sqrt{\omega_0^2 - \sigma^2} = \sqrt{20^2 - 4^2} = 19{,}6 \qquad \sqrt{20^2 - 4^2} = 19{,}6 \quad \text{Rad/s}$$

$$R = \frac{1}{2 \cdot j \cdot m \cdot \omega_d} \qquad = \frac{1}{2 \cdot j \cdot 1.000 \cdot 19{,}6}$$

$$= \frac{1}{2 \cdot j \cdot 19{,}6} = -j2{,}55E - 02 \qquad = -j2{,}55E - 05 \text{ m/N} \cdot \text{s}$$

Log |H|

$$\frac{R}{\sigma} = 6{,}38E - 03 \ \frac{m}{N}$$

$$6{,}38E - 06 \ \frac{m}{N}$$

19,6 Rad.s⁻¹

h(τ)

0,051e⁻ᵟᵗ

0,00051e⁻ᵟᵗ

Time

∠H

0

-90

-180

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The DOF and Multiple-degree-of-freedom (MDOF) Models

Previous models have been restricted to the SDOF case, with only one movement in a single direction. Real structures have many points which can move independently - many degrees-of-freedom. To make an FRF measurement on a real structure we have to measure the excitation and response between two points. But any point may have up to six possible ways of moving so we must also specify the measurement direction.

● **A degree-of-freedom (DOF)** is a *measurement-point-and-direction* defined on a structure. An index i is used to indicate a response DOF, and j an excitation DOF. Additional indices x,y and z may be used to indicate the direction.
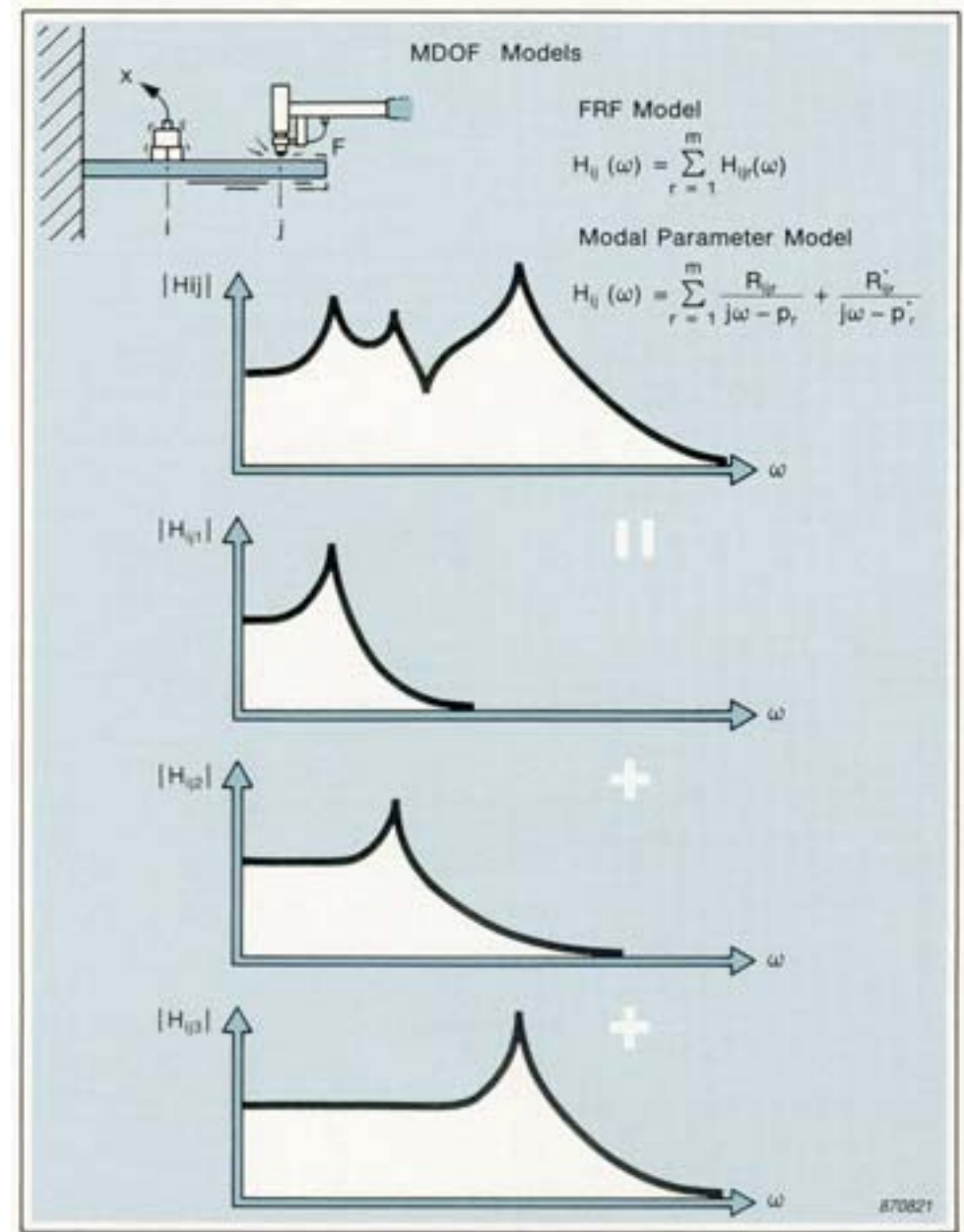
Thus

$$Hij(\omega) \equiv \frac{X_i(\omega)}{F_j(\omega)}$$

By writing $H_{ij}(\omega)$ in two different ways, we obtain the two MDOF models shown as equations in the illustration.

● **The MDOF FRF-model** represents $H_{ij}(\omega)$ as the sum of SDOF FRFs, one for each mode within the frequency range of the measurement, where r is the mode number and m is the number of modes in the model.

● **The MDOF modal-parameter model** defines $H_{ij}(\omega)$ in terms of the pole locations and residues of the individual modes. This model indicates two significant properties of the modal parameters:

• **Modal frequency and damping are global properties.**
The pole location has only a mode number (r) and is independent of the DOFs used for the measurement.

• **The residue is a local property.** The index (ijr) relates it to a particular combination of DOFs and a particular mode.

MDOF Models

FRF Model

$$H_{ij}(\omega) = \sum_{r=1}^{m} H_{ijr}(\omega)$$

Modal Parameter Model

$$H_{ij}(\omega) = \sum_{r=1}^{m} \frac{R_{ijr}}{j\omega - p_r} + \frac{R_{ijr}^*}{j\omega - p_r^*}$$

$|Hij|$

$|H_{ij1}|$

$|H_{ij2}|$

$|H_{ij3}|$

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What is a Mode Shape?

A mode shape is, as we said in the bell example on page 4, a deflection-pattern associated with a particular modal frequency - or pole location. It is neither tangible nor easy to observe. It is an abstract mathematical parameter which defines a deflection pattern as if that mode existed in isolation from all others in the structure.

The actual physical displacement, at any point, will always be a combination of all the mode shapes of the structure. With harmonic excitation close to a modal frequency, 95% of the displacement may be due to that particular mode shape, but random excitation tends to produce an arbitrary "shuffling" of contributions from all the mode shapes.
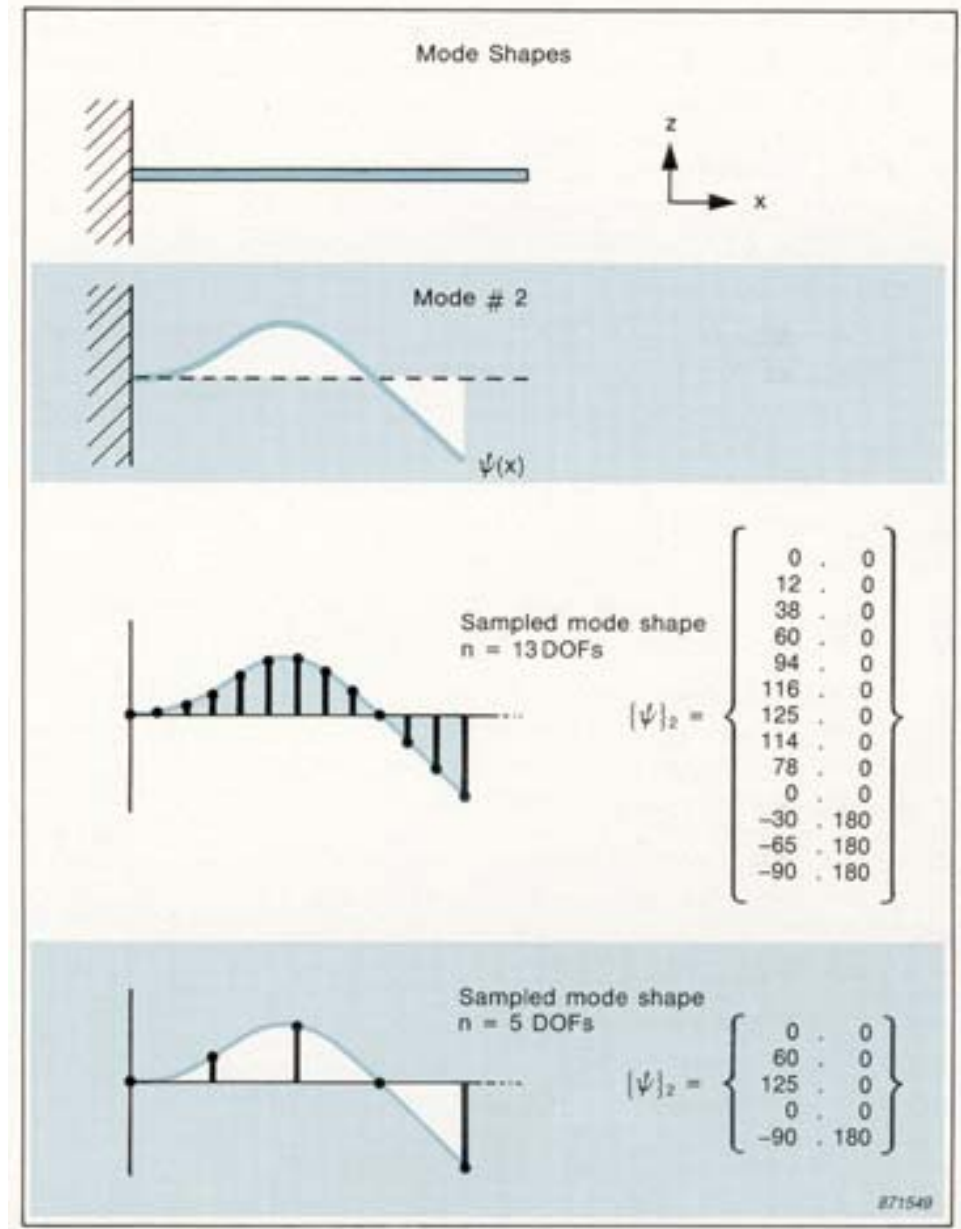
Nevertheless, a mode shape is an inherent dynamic property of a structure in "free" vibration (when no external forces are acting). It represents the *relative displacements* of all parts of the structure for that particular mode.

• Sampled mode shapes → the mode shape vector

Mode shapes are continuous functions which, in modal analysis, are sampled with a "spatial resolution" depending on the number of DOFs used. In general they are not measured directly, but determined from a set of FRF measurements made between the DOFs. A sampled mode shape is represented by the mode shape vector $\{\Psi\}_r$, where r is the mode number.

• Modal displacement

The elements Ψ_{ir} of the mode shape vector are the relative displacements of each DOF (i). They are usually complex numbers describing both the magnitude and phase of the displacement.

Mode Shapes

Mode # 2

$\psi(x)$

Sampled mode shape
n = 13 DOFs

$$\{\psi\}_2 = \begin{Bmatrix} 0 & . & 0 \\ 12 & . & 0 \\ 38 & . & 0 \\ 60 & . & 0 \\ 94 & . & 0 \\ 116 & . & 0 \\ 125 & . & 0 \\ 114 & . & 0 \\ 78 & . & 0 \\ 0 & . & 0 \\ -30 & . & 180 \\ -65 & . & 180 \\ -90 & . & 180 \end{Bmatrix}$$

Sampled mode shape
n = 5 DOFs

$$\{\psi\}_2 = \begin{Bmatrix} 0 & . & 0 \\ 60 & . & 0 \\ 125 & . & 0 \\ 0 & . & 0 \\ -90 & . & 180 \end{Bmatrix}$$

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Normal Modes and Complex Modes

Modes can be divided into two classes

• Normal modes

These are characterized by the fact that all parts of the structure are moving either in phase, or 180° out of phase, with each other. The modal displacements Ψ_{ir} *are* therefore real and are positive or negative. Normal mode shapes can be thought of as standing waves with fixed node lines.

• Complex modes

Complex modes can have any phase relationship between different parts of the structure. The modal displacements Ψ_{ir} are complex and can have any phase value. Complex mode shapes can be considered as propagating waves with no stationary node lines.

• Where to expect normal/complex modes

The damping distribution in a structure determines whether the modes will be normal or complex. When a structure has very light or no damping it exhibits normal modes. If the damping is distributed in the same way as inertia and stiffness (proportional damping), we can also expect to find normal modes.

Structures with very localized damping, such as automobile bodies with spot welds and shock absorbers, have complex modes.

Warning. The mode shapes derived from poor measurements can indicate complex modes on structures where normal modes exist.

Normal Mode

(Standing wave)   Node point

Complex Mode

(Propagating wave)

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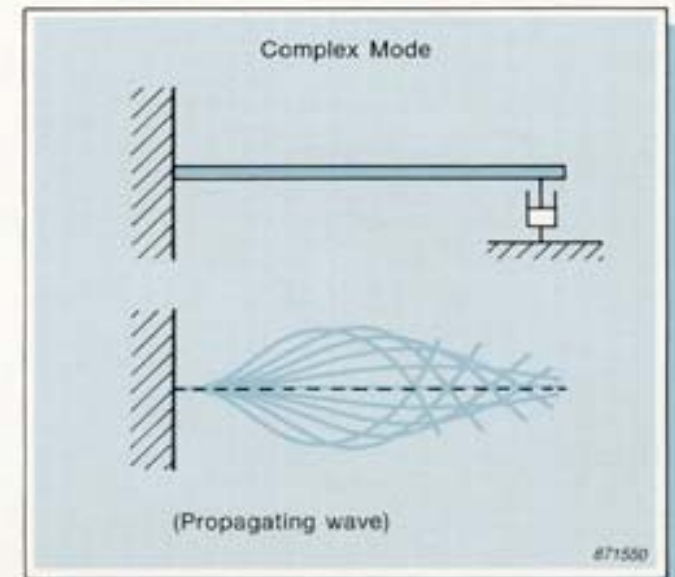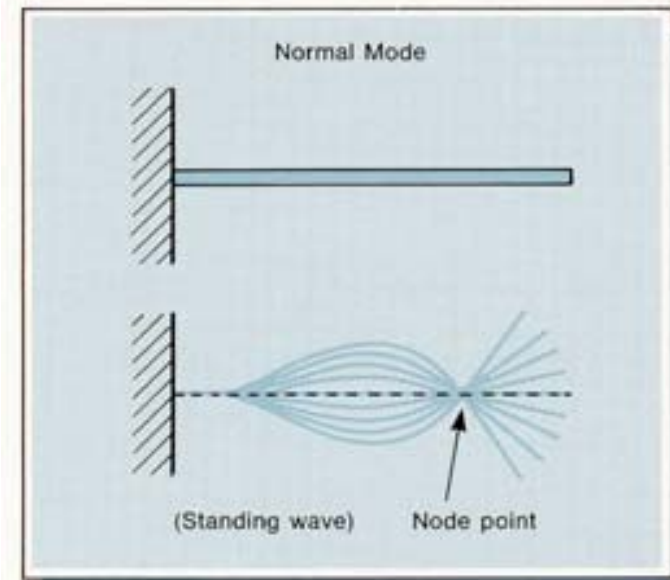
How Residues Relate to Mode Shapes

On page 9 we saw that the residue is proportional to the magnitude of the FRF. At a modal frequency (ω_{dr}) the magnitude is:

$$H_{ij}(\omega) \Big|_{\omega = \omega_{dr}} \approx \frac{R_{ijr}}{\sigma_r}$$

It can be shown that the residue for a particular mode (r) is proportional to the product of the modal displacement Ψ_{ir} at the response DOF, and Ψ_{jr} at the excitation DOF.

$$R_{ijr} \propto \psi_{ir} \cdot \psi_{jr}$$

The illustration shows the second mode of a cantilever beam, with excitation applied at DOF 8, and responses measured at three DOFs. Notice that the shape of the resonance curve is the same for each measurement, but that the magnitude is proportional to the modal displacements.

Residues versus Mode Shape

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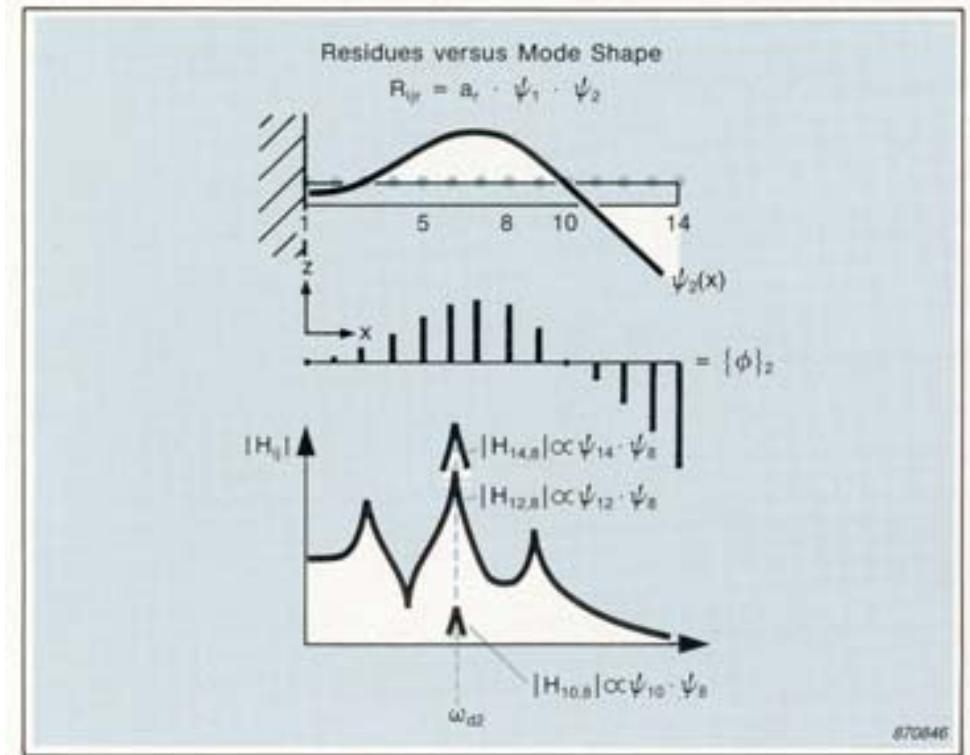
Scaling the Mode Shapes

The mode shape vector $\{\Psi\}_r$ defines the *relative* displacement of each DOF, the values of the vector elements Ψ_{ir} are not unique.

From FRF measurements we determine the residues, which do have unique values. The relationship between a residue and the associated modal displacements, allows us to determine a scaling constant a_r for each mode, such that:

$$R_{ijr} = a_r \cdot \phi_{ir} \cdot \phi_{jr}$$

where ϕ and $_j\phi$ are the *scaled* modal displacements. For a driving-point measurement, this gives us:

$$R_{jjr} = a_r \cdot \phi_{jr}^{2}$$

The rigorous mathematical approach to modal analysis establishes a relationship (given in the illustration) between the mode shape vector $\{\phi\}_r$ and the *modal mass* Mr. If we apply this to the SDOF case (in which there is only one displacement and one mass), we can then evaluate a_r.

What is the modal mass? The modal mass is not related to the mass of the structure and cannot be measured. It is simply a mathematical device which can have any value except zero. We can choose its value and then calculate a_r. For simplicity we will work with unit modal mass scaling ($M_r = 1$).

Scaled mode shapes. From a driving-point measurement, we can obtain R_{jjr} for each mode. By using calculated a_r values, we can then obtain the scaled driving-point displacements ϕ_{jr}. From the response measurements, we are then able to scale the values of ϕ_{ir}, and produce scaled mode shapes.

Chose Your Modal Mass

25 50 75 100

Modal Mass Relates Mode Shapes To Mass-distribution

Modal mass	$M_r = \{\phi\}_r^T [m] \{\phi\}_r$
for SDOF	$M = \phi m \phi$
and	$R = \dfrac{1}{2j\omega_d m} = a \cdot \phi \cdot \phi$
hence	$a = \dfrac{1}{2j\omega_d M}$
for MDOF chose	$M_r = 1$ to give $a_r = \dfrac{1}{2j\omega_{dr}}$

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Modal Coupling

Modal coupling is a general term used to indicate how much of the response, at one modal frequency, is influenced by contributions from other modes. It can be observed in a displayed FRF around a modal frequency.
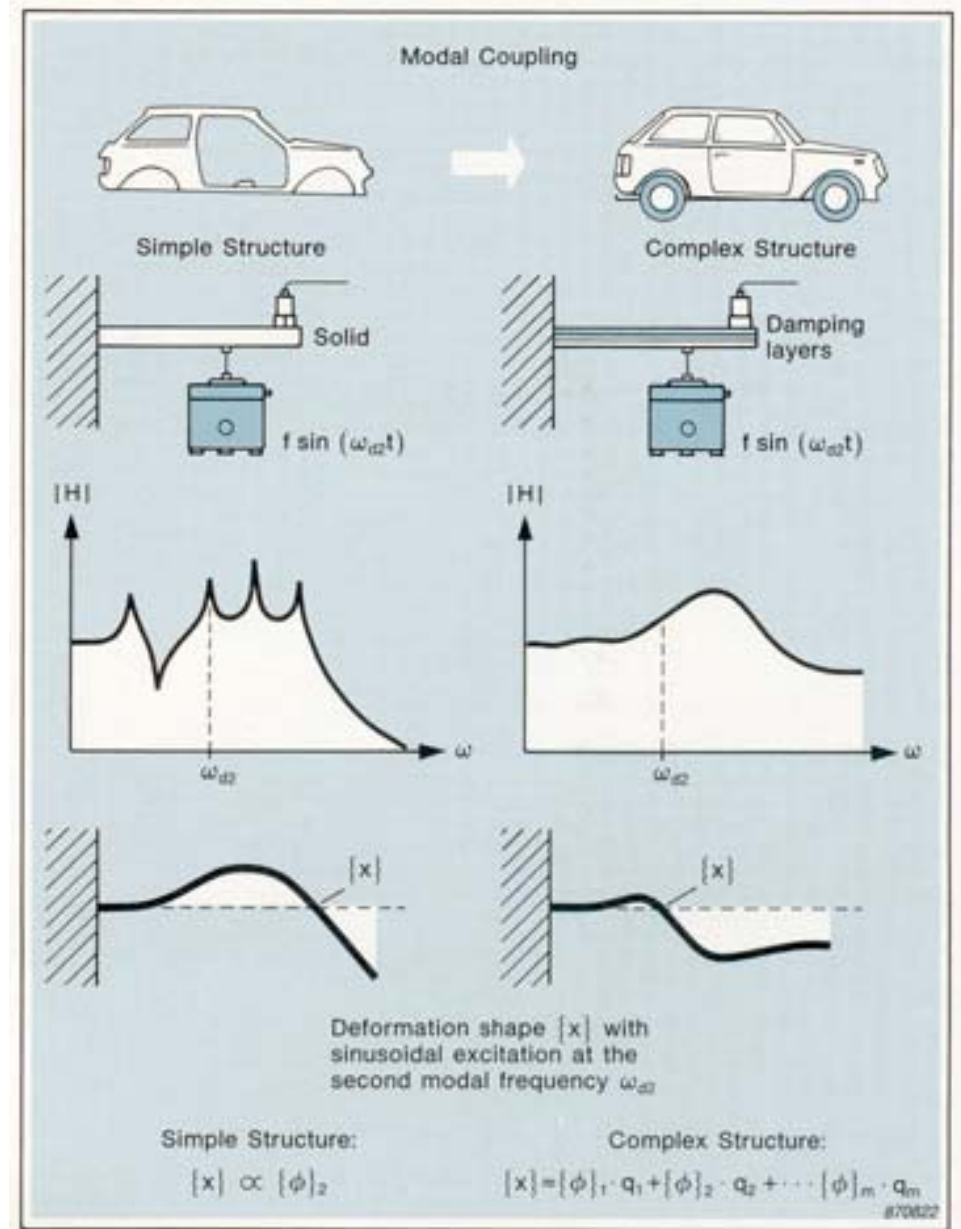
● **Lightly coupled modes - simple structures**

On a lightly damped structure the modes are well separated and distinct and are said to be lightly coupled. Such structures behave as SDOF systems around the modal frequencies, and are known as simple structures.

When testing this type of structure, simple methods give very reliable results. Simple structures are often encountered in trouble-shooting since most noise, vibration and fatigue problems are associated with lightly damped resonances.

● **Heavily coupled modes - complex structures**

On a structure that has heavy damping or high modal density, the FRFs do not display clearly distinctive modes. The modes are said to be heavily coupled and the response at any frequency is a combination of many modes. Complex structures can still be described using a discrete set of modes, but the techniques required to determine the modal parameters are more complicated.

Modal Coupling

Simple Structure

Solid

$f \sin(\omega_{d2} t)$

Complex Structure

Damping layers

$f \sin(\omega_{d2} t)$

$|H|$

ω_{d2}

ω

$|H|$

ω_{d2}

ω

$\{x\}$

$\{x\}$

Deformation shape $\{x\}$ with sinusoidal excitation at the second modal frequency ω_{d2}

Simple Structure:

$$\{x\} \propto \{\phi\}_2$$

Complex Structure:

$$\{x\} = \{\phi\}_1 \cdot q_1 + \{\phi\}_2 \cdot q_2 + \cdots \{\phi\}_m \cdot q_m$$

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# What Does the Modal Description Assume?

On the previous page we looked at the implications of high modal density and heavy damping. But neither of these two factors will prevent us from applying a modal description to a structure. They merely complicate the techniques required.

An assumption that we do have to make, however, is linearity.

● **Linearity**

We have to assume that the systems we test behave linearly so that the response is always proportional to the excitation. This assumption has three implications for Frequency Response Function (FRF) measurements.

- **Superposition.** A measured FRF is not dependent on the type of excitation waveform used. A swept sinusoid will give the same result as a broadband excitation.

- **Homogeneity.** A measured FRF is independent of the excitation level.

- **Reciprocity.** In a linear mechanical system a particular symmetry exists which is described by Maxwell's Reciprocity Theorem. This implies that the FRF measured between any two DOFs is independent of which of them is used for excitation or response.

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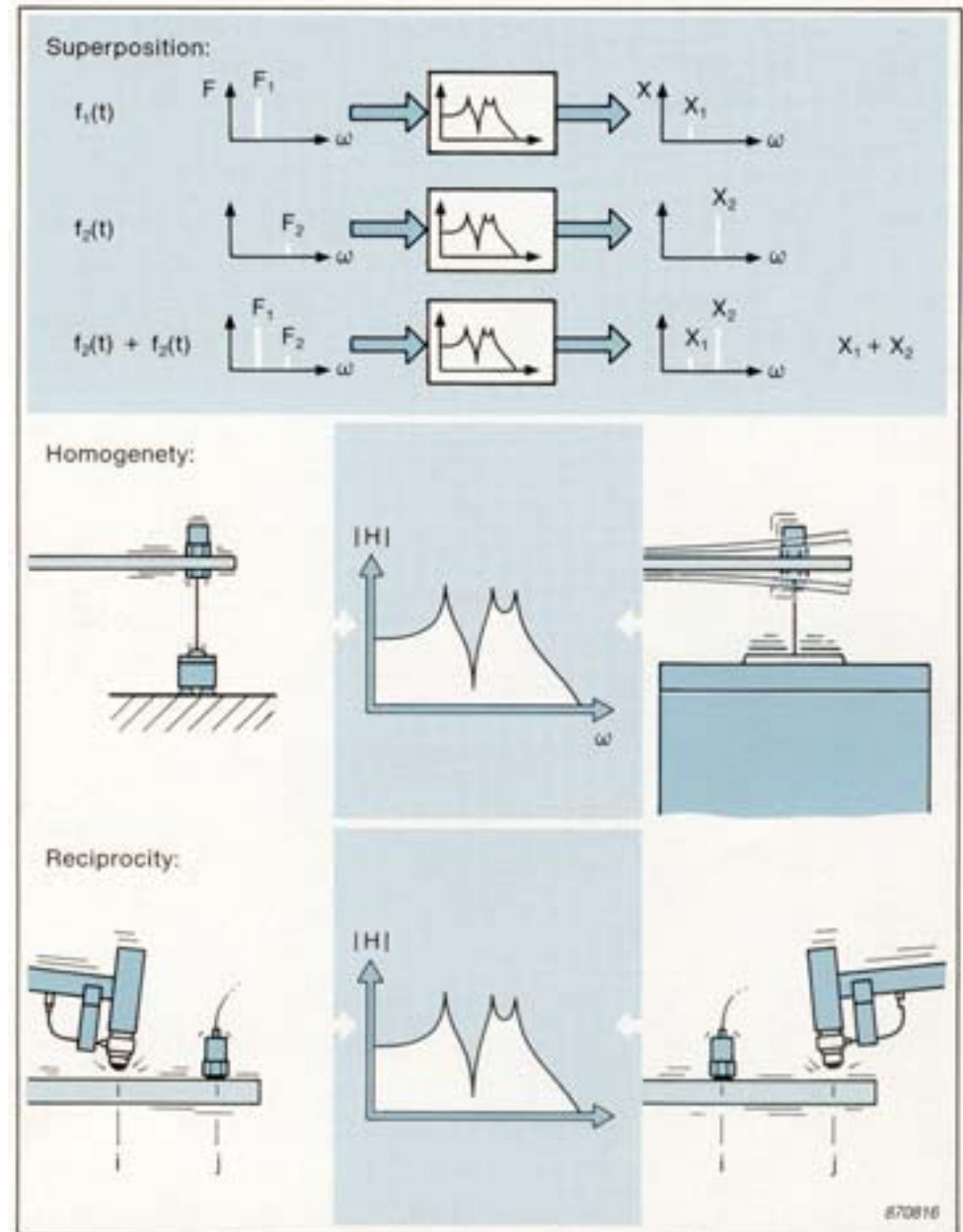
Practical Structures

In general, structures will behave linearly for small deflections. But linearity is often violated when deflections become large, so a modal description cannot be used to predict catastrophic failure.

We must also assume that our structures are:

- **Causal.** They will not start to vibrate before they are excited.

- **Stable.** The vibrations will die out when the excitation is removed.

- **Time-invariant.** The dynamic characteristics will not change during the measurements.

Note. The characteristics of some structures will change during a test:

The characteristics of a lightweight structure may be changed by transducer loading.

Over long tests periods, structural characteristics may be altered by temperature, or other environmental changes.

Some structures may be changing continuously. The mass of a flying aircraft, for example, will decrease gradually as fuel is burned.

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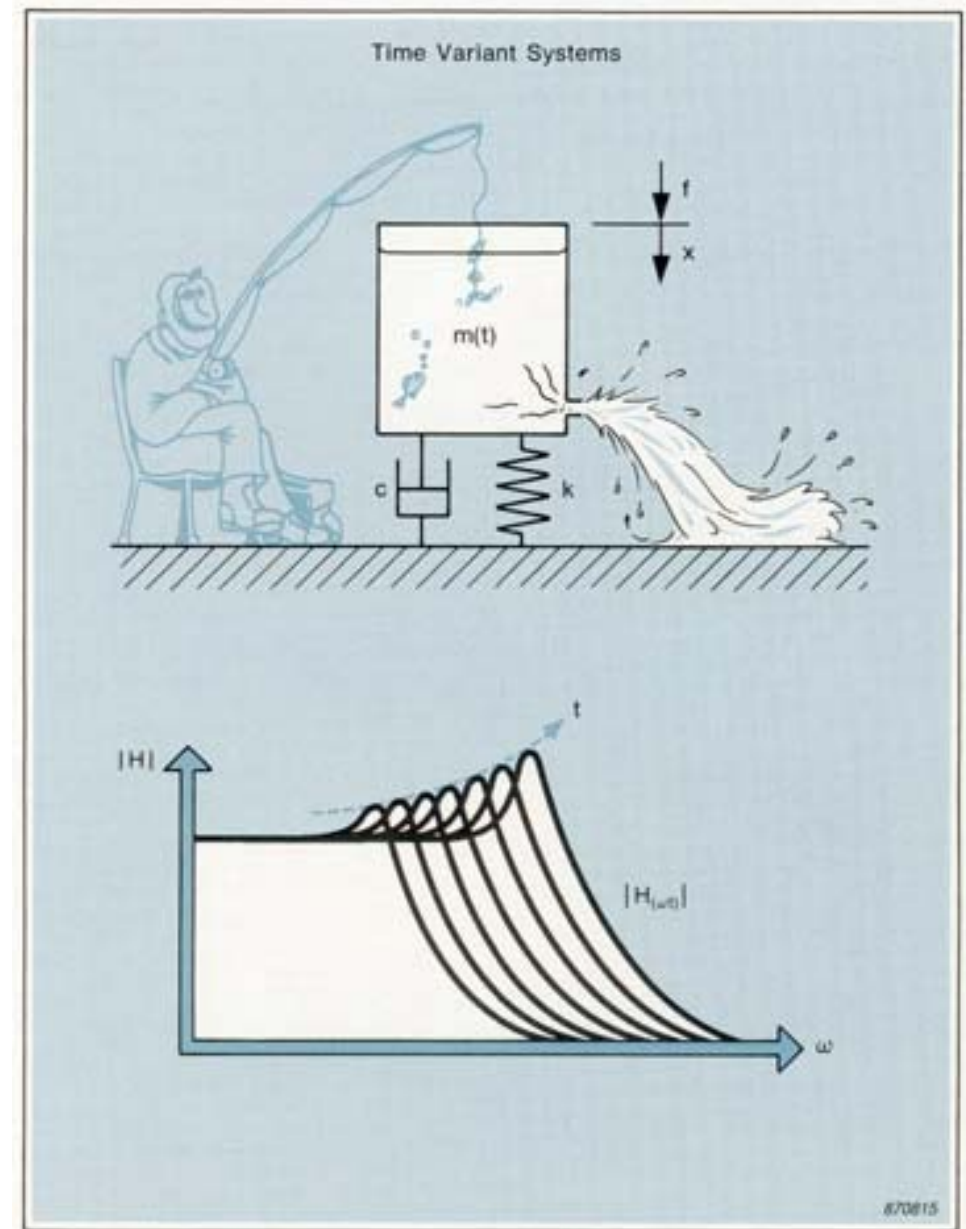
The Lumped-parameter Model and Modal Theory

Much of the theory of modal analysis is based on the mathematics of vectors and matrices. In this primer we do not intend to follow the rigorous mathematical route, but in order to to understand why modal analysis techniques are valid we need to look at a few theoretical points.
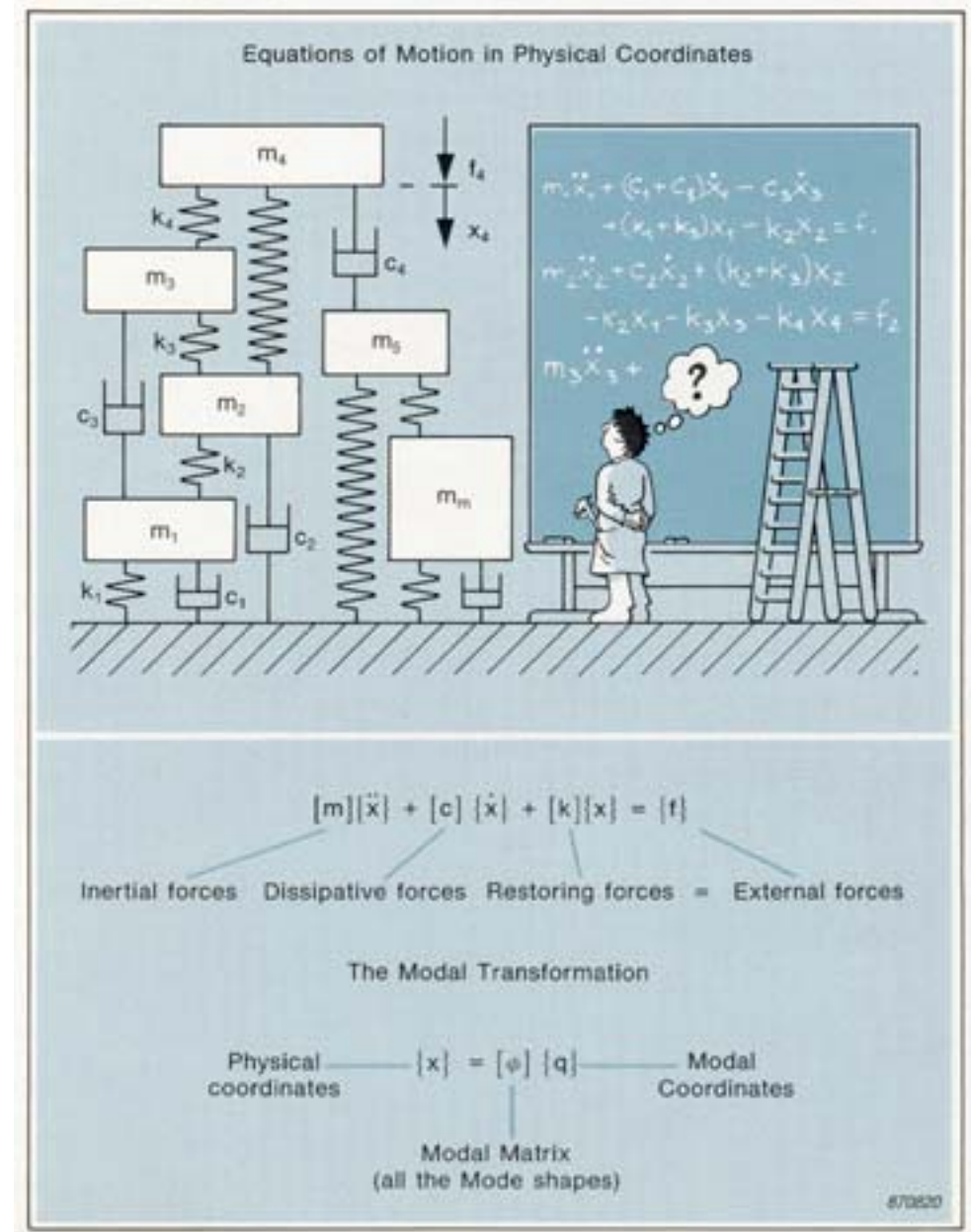
● **The lumped-parameter model**

This model represents an MDOF structure as a series of masses, connected together by springs and dampers. By applying Newton's Second Law we can generate a series of equations for the motion, one equation for each mass (each degree of freedom) in the model.

The mathematical way to organize these equations is to use matrix notation. The mass matrix will contain single mass values, but the damping and stiffness matrices will have combinations of values which couple all the equations together. This coupling indicates that a force applied to one mass will cause a reaction in all the others, making analysis of this model complicated.

On a real structure, the mass, damping and stiffness distributions will not usually be known, but we can measure the pole locations (damping and modal frequencies) and the residues, and obtain the scaled mode shapes. With these parameters we can transform the lumped-parameter model

● **The modal transformation**

If we replace the physical coordinates, in the (matrix) equation of motion, with the product of the modal matrix (all the scaled mode shape vectors as columns) and the modal coordinates, we make a transformation into another domain - the modal space.

Equations of Motion in Physical Coordinates

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f\}$$

Inertial forces   Dissipative forces   Restoring forces   =   External forces

The Modal Transformation

Physical coordinates —— $\{x\} = [\phi]\{q\}$ —— Modal Coordinates

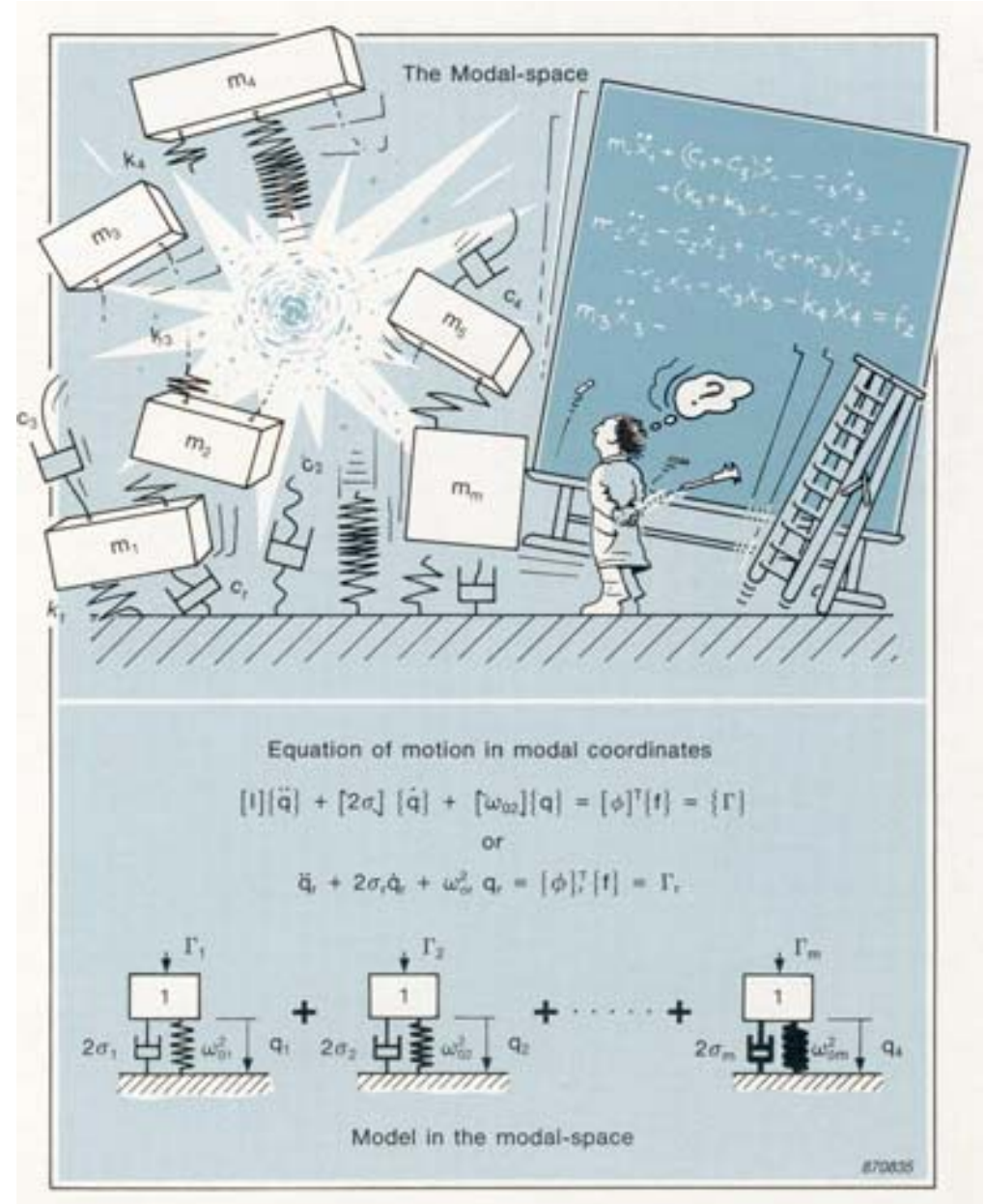Modal Matrix
(all the Mode shapes)

The Modal Space

Transformation into the modal space has a dramatic effect on the lumped-parameter model. The equations of motion become decoupled, and can be seen as a collection of independent SDOF models, one for each mode in the MDOF model (each modal coordinate).

Each model has a mass of unity (unit modal-mass), a damping constant equal to the bandwidth of the mode, and a spring constant equal to the square of the the undamped natural frequency. The individual models are excited by a modal force equal to the dot (scalar) product of the mode shape and the physical force vector (that is, the projection of the force on the mode shape). The modal force can be interpreted as the ability of a specific force distribution to excite a particular mode.

We can now write a set of equations, in terms of the modal parameters, with solutions in modal coordinates. The equation for each coordinate can be solved independently as an SDOF system. *When the mode shapes are scaled with unit modal mass, the equations are in terms of the simple, measurable parameters of natural frequency, modal damping and scaled mode shape.*

The Modal-space

Equation of motion in modal coordinates

$$[I]\{\ddot{q}\} + [2\sigma]\{\dot{q}\} + [\omega_{0}^2]\{q\} = [\phi]^T\{f\} = \{\Gamma\}$$

or

$$\ddot{q}_r + 2\sigma_r\dot{q}_r + \omega_{0r}^2 q_r = [\phi]_r^T\{f\} = \Gamma_r$$

Model in the modal-space

Specifying the Degrees of Freedom (DOFs)

A free point generally has six degrees-of-freedom. Three translational, and three rotational. No suitable rotational-transducers are available, but the translational degrees of freedom are usually sufficient to describe the motion. For most practical structures, a set of regularly distributed measurement points in one or two directions is adequate.

● **How many DOFs are needed for a test?**

The number of DOFs required depends on the purpose of the test, on the structural geometry, and on the number of modes in the frequency range of interest.

A test made simply to verify analytically predicted modal frequencies requires only a few DOFs.

If the purpose of the test is to construct a mathematical model, then sufficient DOFs must be used for the measured mode shapes to be mutually orthogonal, or linearly independent.

The illustration shows two examples of measurements made on a rectangular plate, one using four and the other using thirty DOFs. In the four DOF example, a maximum of four linearly independent modes are seen. The higher modes are simply the first four repeated. A model based on this measurement could only be used up to a frequency which included the first three or four modes. In the thirty DOF example, the shapes for the two highest modes are only roughly represented.

Note. The number of DOFs has to be chosen to represent the total dynamics of the structure. It is the geometrical complexity of the mode shapes, rather than to the number of modes expected, which determines the number of DOFs required.

Degrees-of-Freedom

Measurement of $H_{4z,8y}$
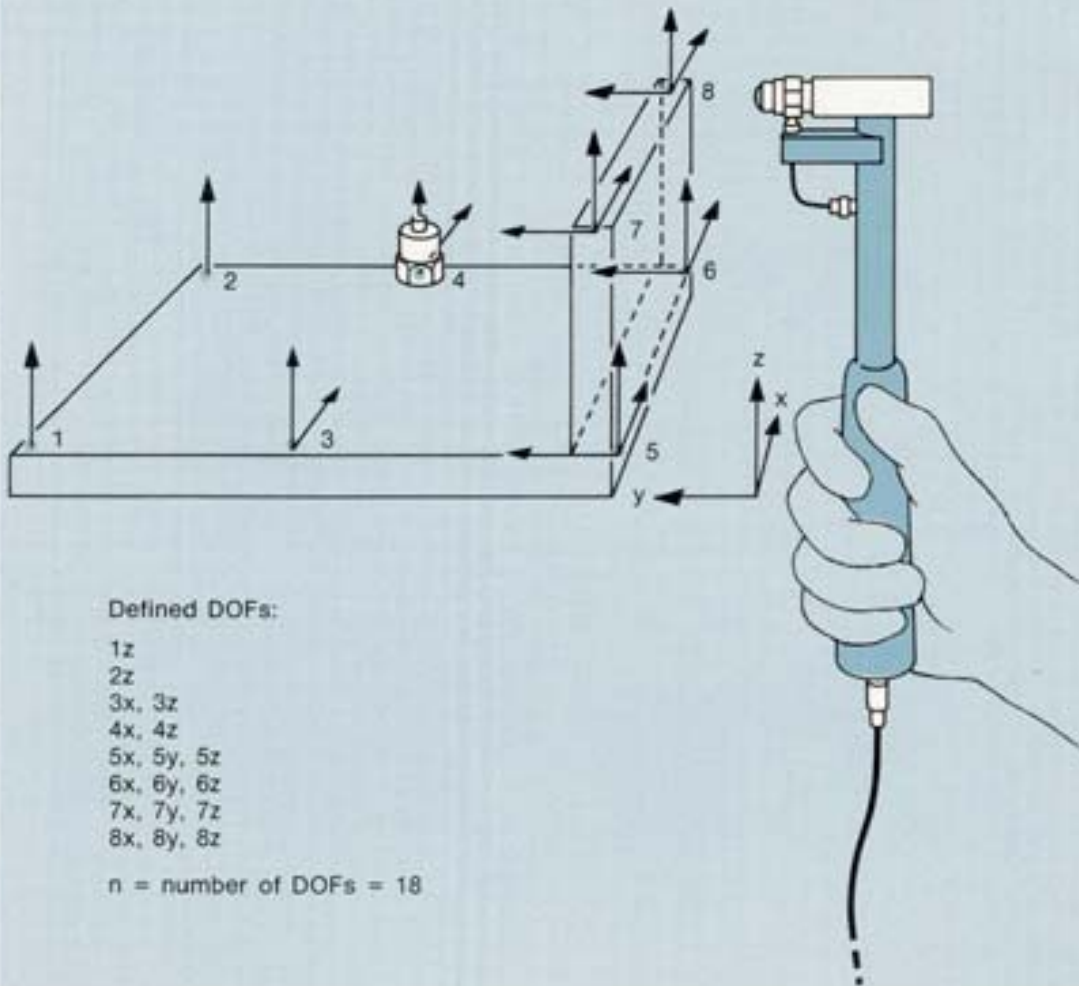
Defined DOFs:

1z
2z
3x, 3z
4x, 4z
5x, 5y, 5z
6x, 6y, 6z
7x, 7y, 7z
8x, 8y, 8z

n = number of DOFs = 18

Mode	30 DOFs		4 DOFs
# 1:		—	
# 2:		—	
# 3:		—	
# 4:		—	
# 5:		—	
# 6:		—	
# 7:		—	
# 8:		—	
# 9:		—	
#10:		—	
#11:		—	
#12:		—	

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DOFs and the Mobility Matrix

● **Input/output combinations**

The illustration shows a structure with two *defined* DOFs, each of which is an input/output point for an FRF measurement. If we have n defined DOFs, the number of possible input/output combinations is n x n.

● **The mobility matrix**

The individual FRF measurements can be arranged as the elements of a matrix, known as the mobility matrix [H]. Each element $H_{ij}(\omega)$ is a particular FRF measurement.

Each row of the matrix contains FRFs with a common response DOF. while in each column they have a common excitation DOF. The diagonal of [H] contains a class of FRFs for which the response and excitation DOFs are the same. These are the *driving point FRFs.* The off-diagonal elements are *transfer FRFs.*

Note. The term mobility is used in a general sense and may represent compliance, mobility or accelerance. In models, [H] generally refers to compliance, but measurements are usually made in terms of accelerance (see Part 1 *Mechanical Mobility Measurements*).
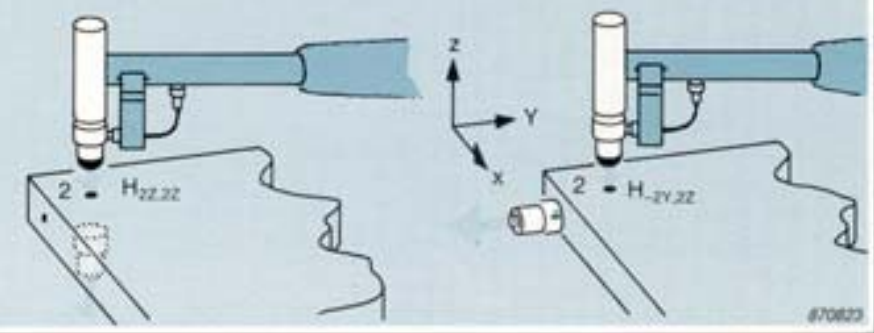
● **Minimum sufficient data**

The number of DOFs specified in a test can range from ten to several hundred. The matrix [H] can thus become enormous (when n = 100, [H] contains 10000 FRFs).

Fortunately, reciprocity helps here and all the information for a linear mechanical structure is contained in either one complete row, or one complete column of [H]. The number of measurements needed is therefore equal to the number of specified DOFs.

Input/Output Combinations for a 2-DOF System

$H_{1Z,1Z}$ $H_{1Z,2Z}$ $H_{2Z,1Z}$ $H_{2Z,2Z}$
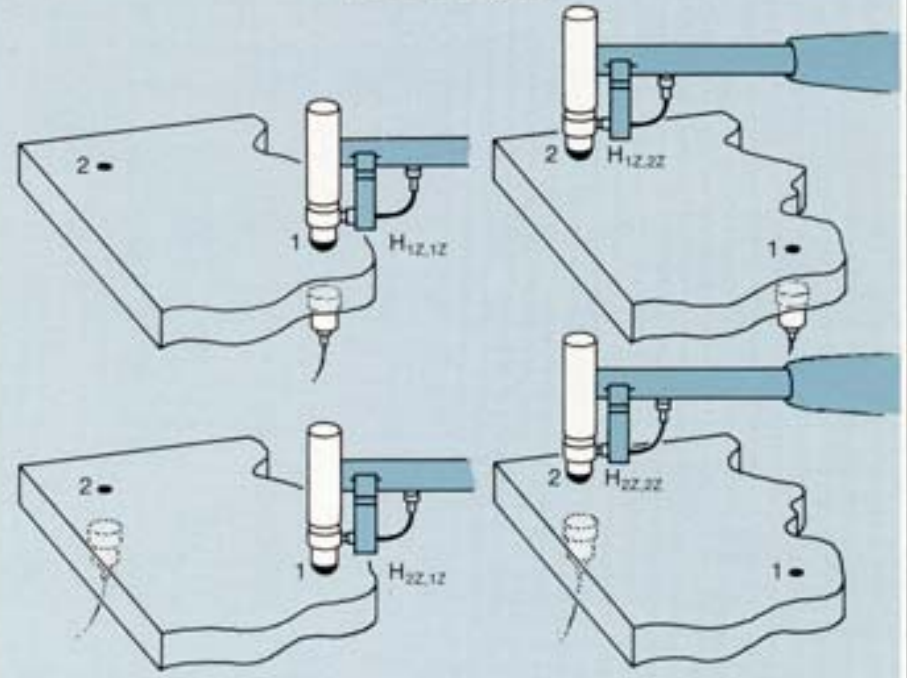
The Mobility Matrix: $[H] = \begin{bmatrix} H_{1Z,1Z} & H_{1Z,2Z} \\ H_{2Z,1Z} & H_{2Z,2Z} \end{bmatrix}$

$H_{2Z,2Z}$ $H_{-2Y,2Z}$

Modal Test on a Simple Structure

To examine the techniques used to extract the modal parameters, let us look at a modal test on a simple structure, an example typical of a trouble-shooting problem.

The example given in the illustration can be considered as a cantilever beam. The simplest instrumentation we can use is a dual-channel signal analyzer, with hammer excitation, and an accelerometer to measure the response signal. We will restrict the investigation to the first few bending modes, so that four DOFs aligned in the vertical direction will be sufficient.

Let us assume that the instrumentation has been set up, preliminary adjustments made, and that a few initial measurements were taken to optimize the frequency range, weighting functions and input conditioning.

● **Determination of pole locations**

Any FRF will indicate that the beam has lightly coupled modes. Therefore structure behaves as a single-degree-of-freedom system around its modal frequencies, around which we can assume that all the response is due to that particular mode.

From any measured FRF we can determine the modal frequencies and dampings, and thus obtain the pole locations.
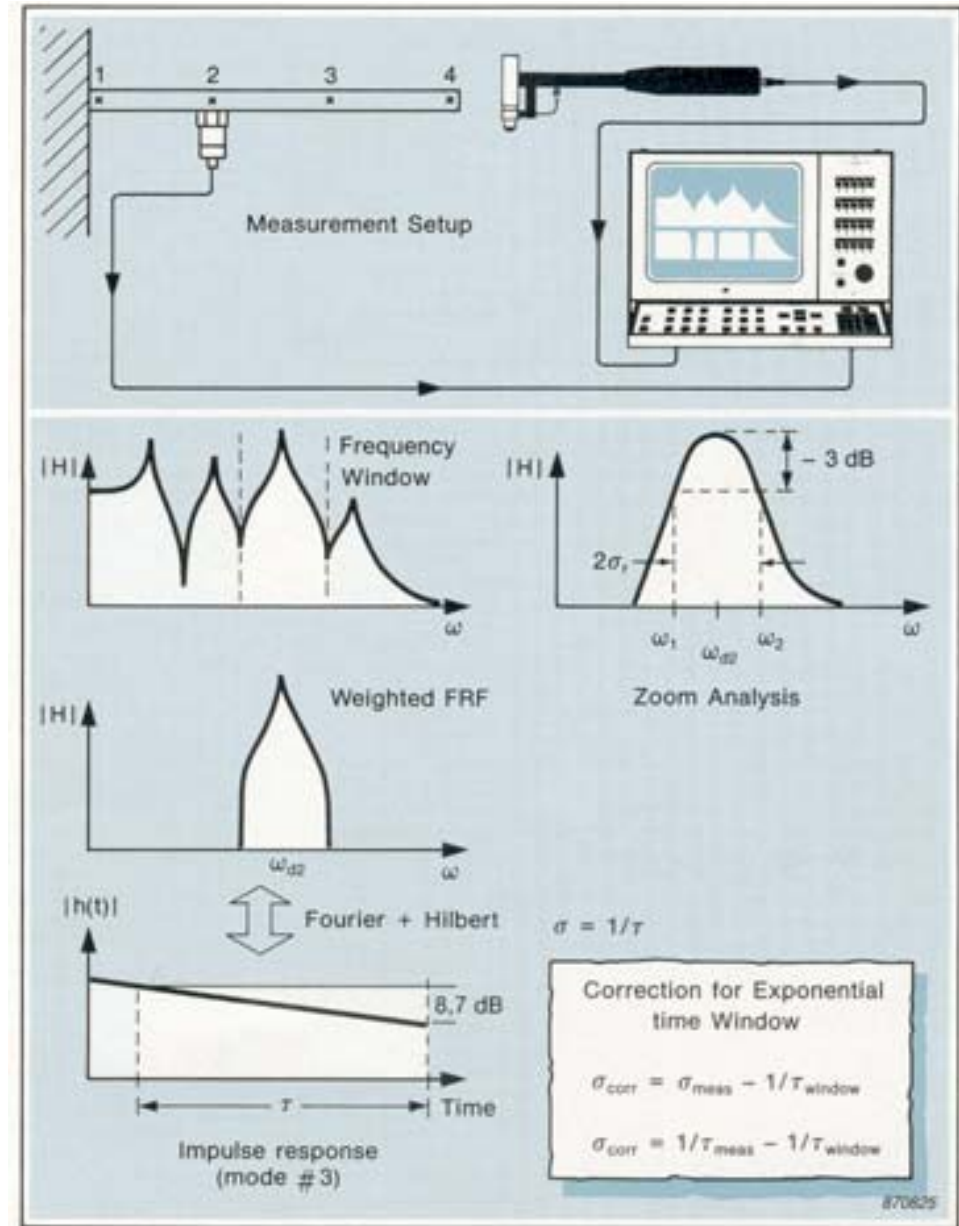

Test on a Simple Structure

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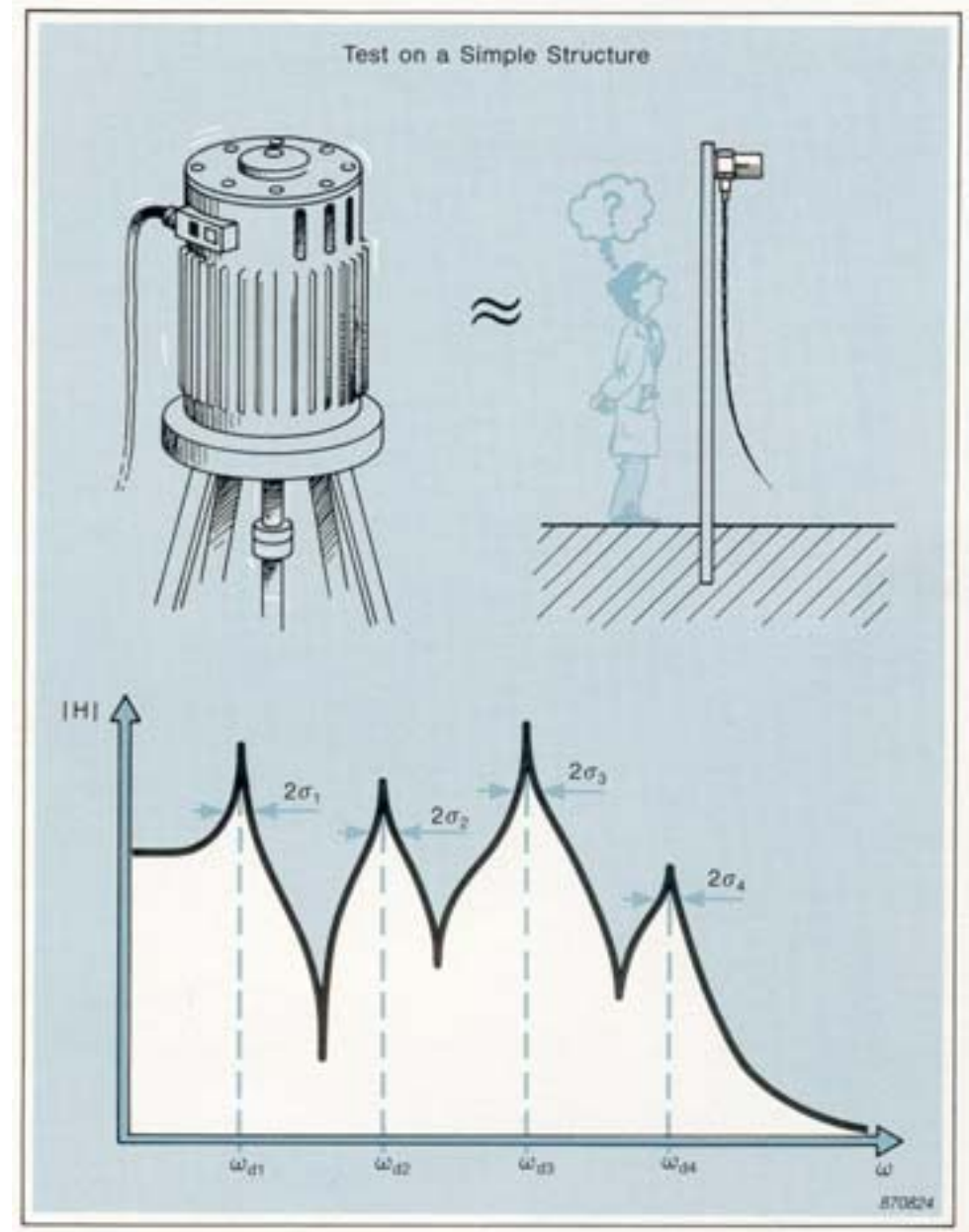
• **The modal frequencies** are determined simply by observing the maximum magnitudes on the FRF.

• **The modal dampings** are not so simple to determine, and will often be the parameters measured with the greatest degree of uncertainty.

One technique we can use to measure the damping is to find the -3dB bandwidths. On a lightly damped structure the resonances are sharp and the peaks are too narrow for accurate measurements of the bandwidths. This problem can often be overcome by making a zoom analysis to obtain sufficient frequency resolution for the measurements.

Alternatively we can use a frequency weighting technique, in which each mode is isolated in turn. Application of both the Fourier and Hilbert transforms will then give the impulse response function of the mode. On a display showing logarithmic magnitude of the impulse response function the decay can be seen as a straight line. From this we can find the time τ taken for the magnitude to decay 8,7 dB. The decay rate σ is the reciprocal of the decay time, so that $\sigma = 1/\tau$.

From these measurements we can find the pole locations, but we also need to determine the associated mode shapes.

Measurement Setup

Weighted FRF

Zoom Analysis

Impulse response
(mode #3)

Correction for Exponential time Window

$$\sigma_{corr} = \sigma_{meas} - 1/\tau_{window}$$

$$\sigma_{corr} = 1/\tau_{meas} - 1/\tau_{window}$$

Mode Shapes from Quadrature Picking

Remember that the equation for the SDOF model, evaluated at one of the modal frequencies ω_{dr}, gives us:

$$H(\omega)\bigg|_{\omega=\omega_{dr}} \approx \frac{R}{\sigma}$$

But since our measurements are in accelerance we need to make a double differentiation to produce an accelerance model:

$$A(\omega)_{dr} \approx \frac{R}{\sigma}(-\omega_{dr}^2)$$

With more than one defined DOF we can substitute the residue by:

$$R = a_r \cdot \phi_{ir} \cdot \phi_{jr} = \frac{\phi_{ir} \cdot \phi_{jr}}{2j\,\omega_{dr}}$$

giving

$$A(\omega)_{dr} \approx j\,\frac{\phi_{ir} \cdot \phi_{jr}}{2\,\sigma} \cdot \omega_{dr}$$
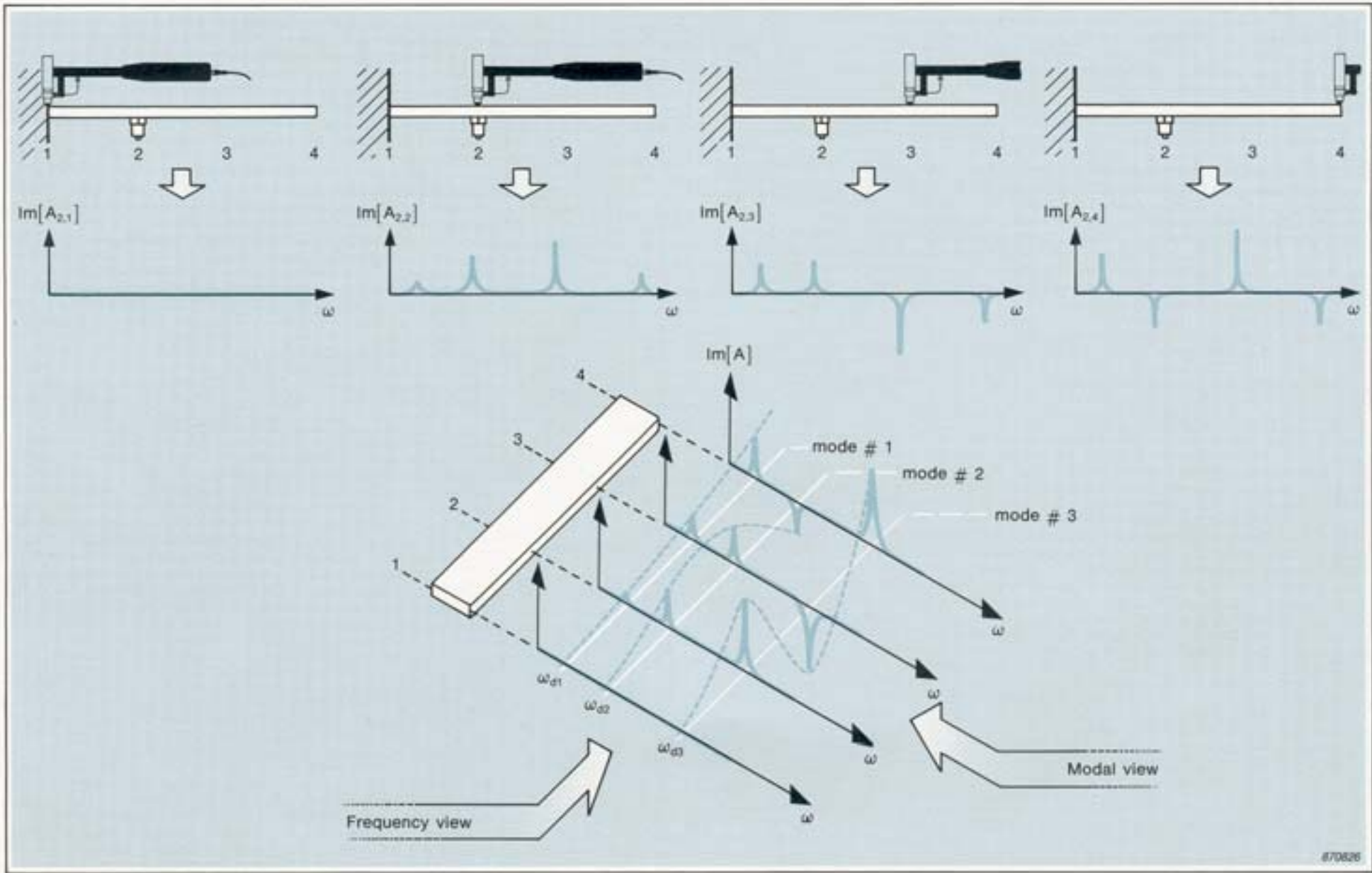
Notice that $A(\omega)$ is an imaginary number at a modal frequency. This gives the basis for the quadrature picking technique, through which we can determine the mode shapes.

The FRF appears to become purely imaginary at the modal frequency. Its amplitude is proportional to the modal displacement, and its sign is positive if displacement is in phase with the excitation.

The mode shapes can be determined if we fix a response, or an excitation DOF as a reference and then make a set of measurements. The imaginary parts of the measured FRFs can be "picked" at the modal frequencies at which they represent the modal displacement for that specific DOF.

In our example DOF # 2 is used as the response reference. A set of FRFs is then measured by exciting each of the four defined DOFs in turn.

The four imaginary parts of the FRF for each modal frequency represent the associated mode shape. If the measurements were made with calibrated instrumentation, the mode shapes could then be scaled.

Im[A_{2,1}]

Im[A_{2,2}]

Im[A_{2,3}]

Im[A_{2,4}]

Im[A]

mode # 1

mode # 2

mode # 3

ω_{d1}

ω_{d2}

ω_{d3}

Frequency view

Modal view

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Parameter Estimation by Curve-fitting

In the previous example the modes were lightly coupled, we determined the modal parameters simply by picking a number of discrete values from the FRF measurements.

When the measured data indicates heavily coupled modes or noise contamination, or when high accuracy is required for the estimation, we can make a computer-aided modal analysis. A curve-fitting technique can then be used to improve the modal parameter estimation.

● **Gold in ⟹ gold out**

Many fine curve-fitting algorithms are available for modal analysis. But whichever parameter-estimation technique we use, *the estimation must always be based on sound data which sufficiently represents the dynamics of the structure:*

- The most important part of modal testing is making the mobility measurements.

- No curve-fitter can make reliable parameter estimates from poor measurements.
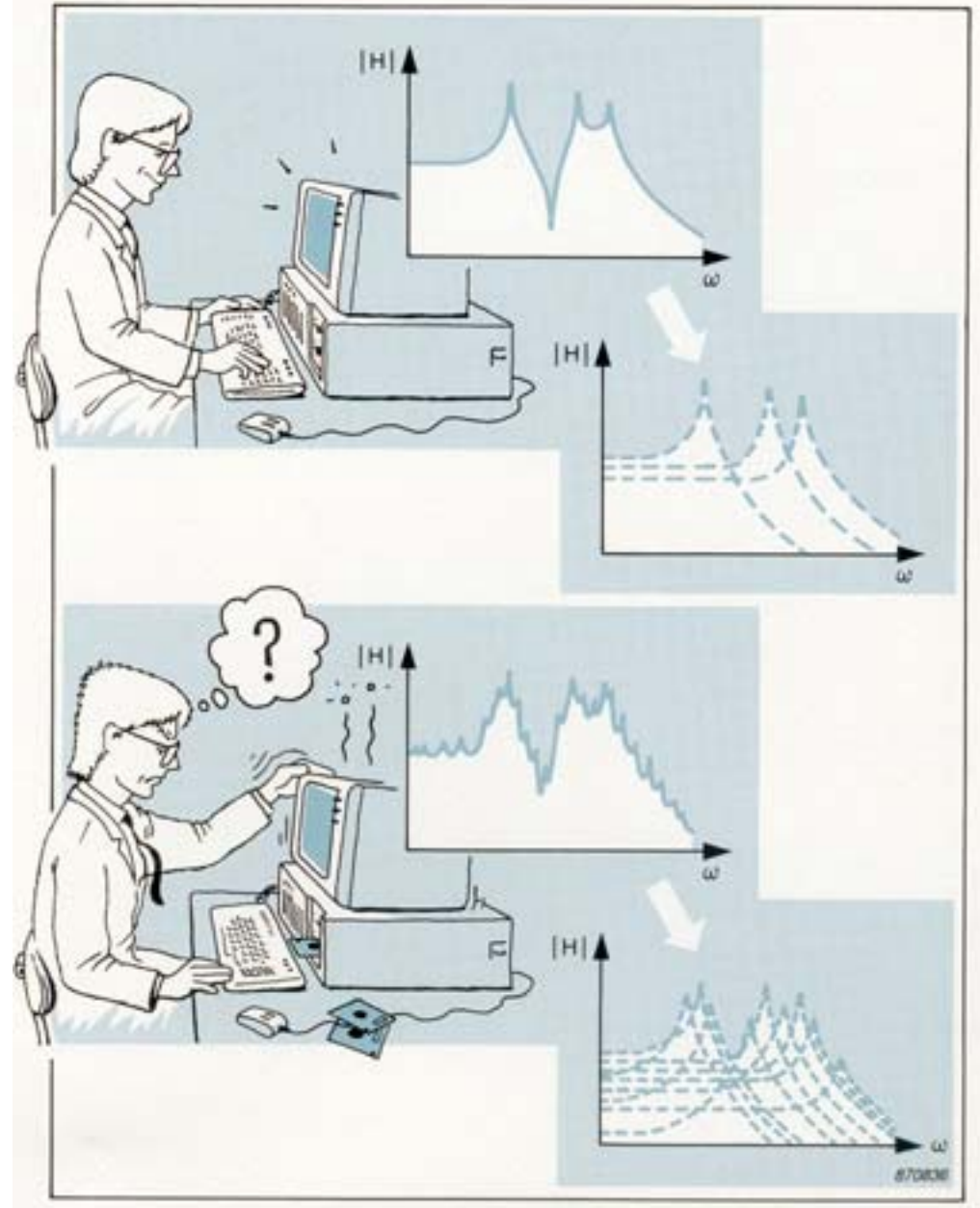
What is Curve-fitting?

Curve-fitting is where the mathematical theory and practical measurements meet. The theory provides us with a mathematical parametric model for the theoretical FRF of a structure, and our measurements give the real FRF. Curve-fitting is the analytical process to determine the mathematical parameters which give the closest possible fit to the measured data.

Let us look at this process by examining an experiment to find the compliance (1/stiffness) of a helical spring. When mass loadings are applied, we can observe and plot the resulting deflections. If we assume a linear relationship between force and deflection, then
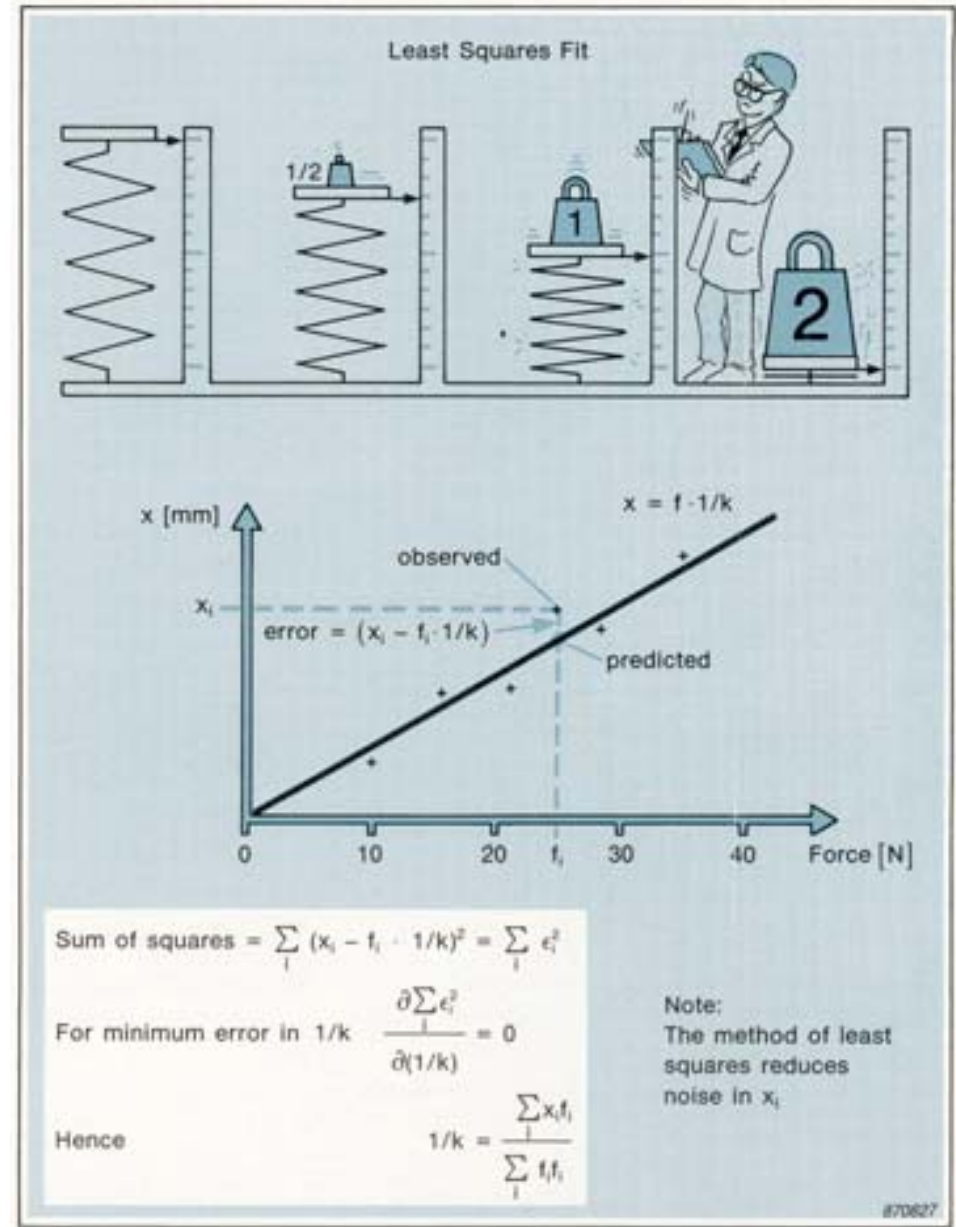
$$x = f \cdot 1/k$$

In this case we have only one unknown (1/k) so one pair of observations would be sufficient. But if we use all the observed data, we will get the best estimate of 1/k.

If the weights are calibrated, then the applied force is accurately known. Any deviation from a straight line can only be due to errors in measuring the deflections (reading noise).

● **Method of least squares**

The method of least squares (MLS), shown in the illustration, is one technique used to minimize the deviation between the measured data and predicted values. It can be used on any mathematical model, including our SDOF and MDOF models.

Least Squares Fit

Sum of squares $= \sum_i (x_i - f_i \cdot 1/k)^2 = \sum_i \epsilon_i^2$

For minimum error in $1/k$ $\dfrac{\partial \sum_i \epsilon_i^2}{\partial (1/k)} = 0$

Hence $1/k = \dfrac{\sum_i x_i f_i}{\sum_i f_i f_i}$

Note:
The method of least squares reduces noise in x_i

Curve-fitters for Modal Analysis

The process of estimating the modal parameters from the measured FRFs is very similar to the previous example.
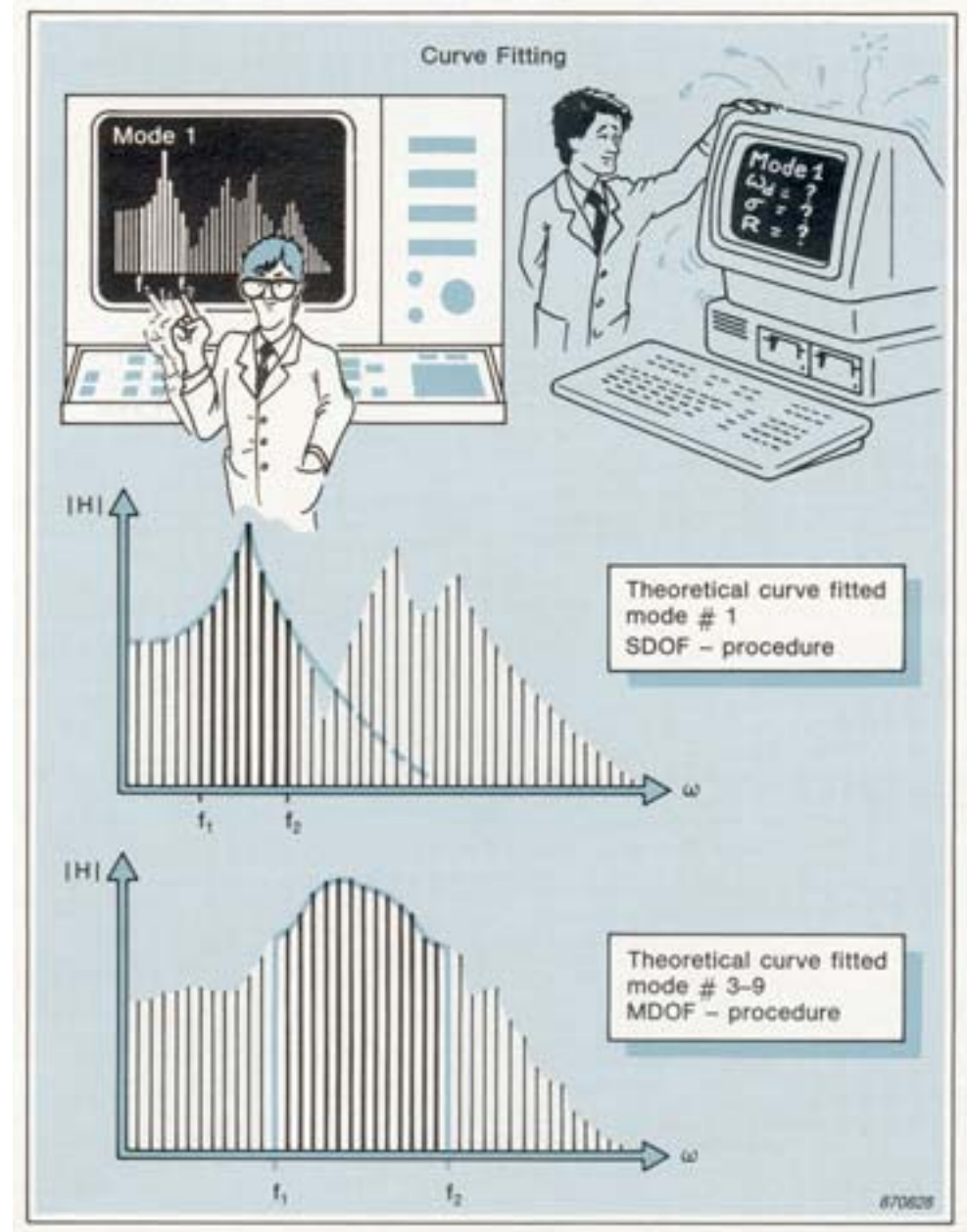
The method of least squares (MLS) can be used to improve the confidence of parameter estimations. For each mode we need to estimate two unknown complex numbers, the pole location and the residue, but for each FRF measurement the dual-channel analyzer gives us around 800 complex values. With this enormous amount of data to be manipulated it is essential to use a computer to implement the estimations. The introduction of MLS curve-fitters takes us away from manual techniques and into computer-aided analysis.

MLS procedures inherently reduce the effects of random noise in the measurements, they smooth the data. They will not generally reduce the effect of bias errors, such as any leakage or phase errors in the measurements, which would still cause erroneous parameter estimations.

Many diverse types of curve-fitters are now available which are outside the scope of this discussion, but some general points are worth noting:

The term "curve-fitting" arises from the general procedure where, after parameter estimation, an analytical curve is generated and superimposed on the measured data so that the operator can evaluate the fit.

Good curve-fitters should be easy to use and have an interactive dialogue with the operator. If there is a choice of curve-fitters, then the simplest one available should be chosen provided that it suits the measured data. This will usually prove to be the best for the application, and the fastest to use.

Curve Fitting

Mode 1

Theoretical curve fitted mode # 1
SDOF – procedure

Theoretical curve fitted mode # 3–9
MDOF – procedure

The aim of curve-fitting is to extract reliable modal data from the measurements. Whilst a well-fitting curve may be necessary, it might not in itself be sufficient. The operator must judge the validity of the procedures.

● **SDOF curve-fitters**

SDOF curve-fitters are used for systems with lightly coupled modes, for which SDOF characteristics can be assumed around the modal frequencies. The operator must decide the frequency span around each modal frequency, over which this assumption can be held. This is always a compromise between including as many data points as possible, to maximize the statistical estimation, and moving so far away from the resonance that other modes become dominant and the SDOF assumption becomes invalid.

● **MDOF curve-fitters**

MDOF curve-fitters are used with heavily coupled modes. The operator has to specify the frequency span over which the curve-fitter will seek the parameters. Some algorithms will always find sufficient modes to fit the curve, but some of them will simply be computational and have to be edited out by the operator. To a large extent, the results will depend on the user's skill and experience in specifying the correct number of modes for the model.

Alternative Curve-fitters
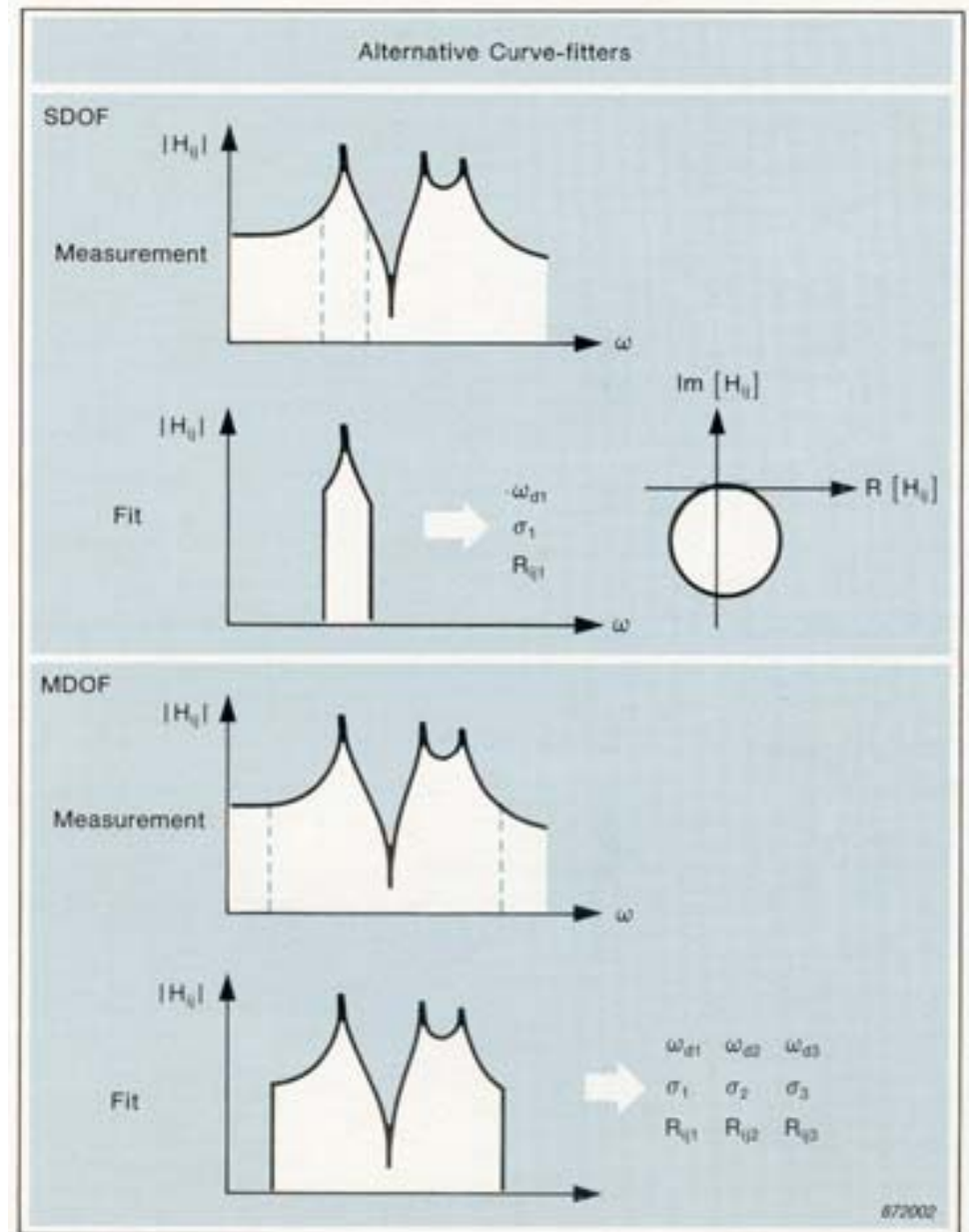
Local and Global Curve-fitters

Curve-fitters can be classified as local or global. This classification depends on how estimations of the modal parameters are made from the data set of FRF measurements.

• Local curve-fitters

Most curve-fitters fall into this category. They fit the pole locations from one, or a few, measurements and fix them for the entire data set. The residues are then fitted for each individual FRF measurement using the locally found pole locations.

The reliability of local curve-fitters depends, in part, on the assumption that locally found pole locations are valid for the entire data set. This assumption may not always be true. Some of the FRF measurements may be difficult to fit because they contain strong local modes. An alternative estimation procedure is provided by global curve-fitters.

• Global curve-fitters

Global curve-fitters estimate the pole location values in a least squares sense from all the measurements in the data set. This procedure enhances the global modes and attenuates purely local modes which may be found in a few FRFs. The curve-fitter then uses the global pole location values to fit the residues for each individual measurement.


Global Curve - fitter
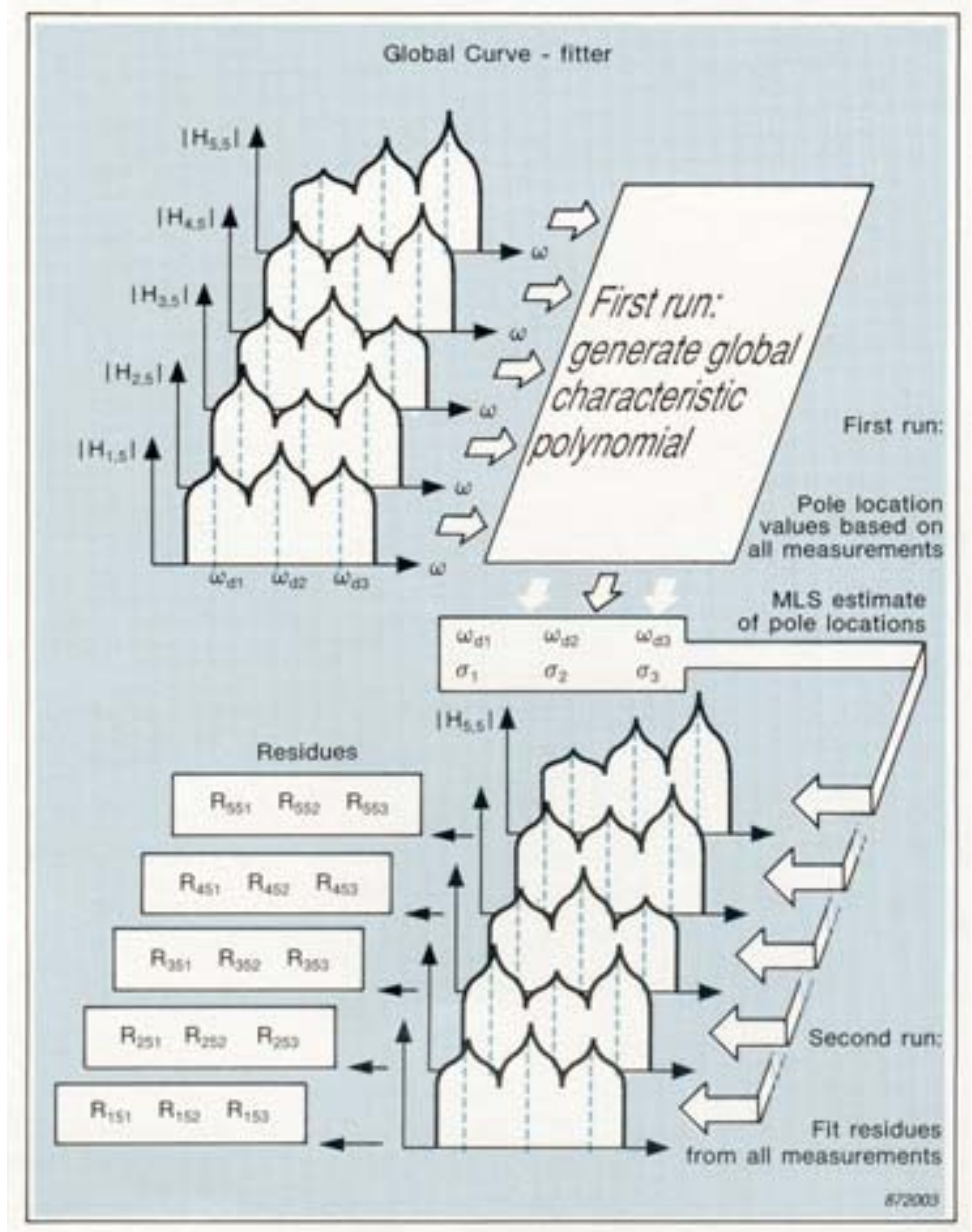
Computer-aided Modal Testing

In many applications of modal analysis, a large number of DOFs have to be defined, and an equivalent number of FRF measurements made. Computer aid is therefore essential for parameter estimation and documentation. Small desktop computers with sufficient power for this application are now available, together with versatile user-friendly programs which guide the user through all the steps in the measurement and analysis procedures.

Modal test on a minibus body

Let us look at a typical computer-aided modal test, and examine each phase of the test in detail.

● **The objectives of the test** are to study the first two elastic vertical modes of a minibus body to verify analytical calculations and to predict its vibration response due to some assumed excitation forces.

● **The test equipment** is a typical Brüel&Kjær structural analysis system with computer-aided modal analysis software.

● **The procedure** can be divided into four steps:

Step 1 - Setting up the modal test

Step 2 - Making the measurements

Step 3 - Parameter estimation by curve-fitting

Step 4 - Documentation of the test
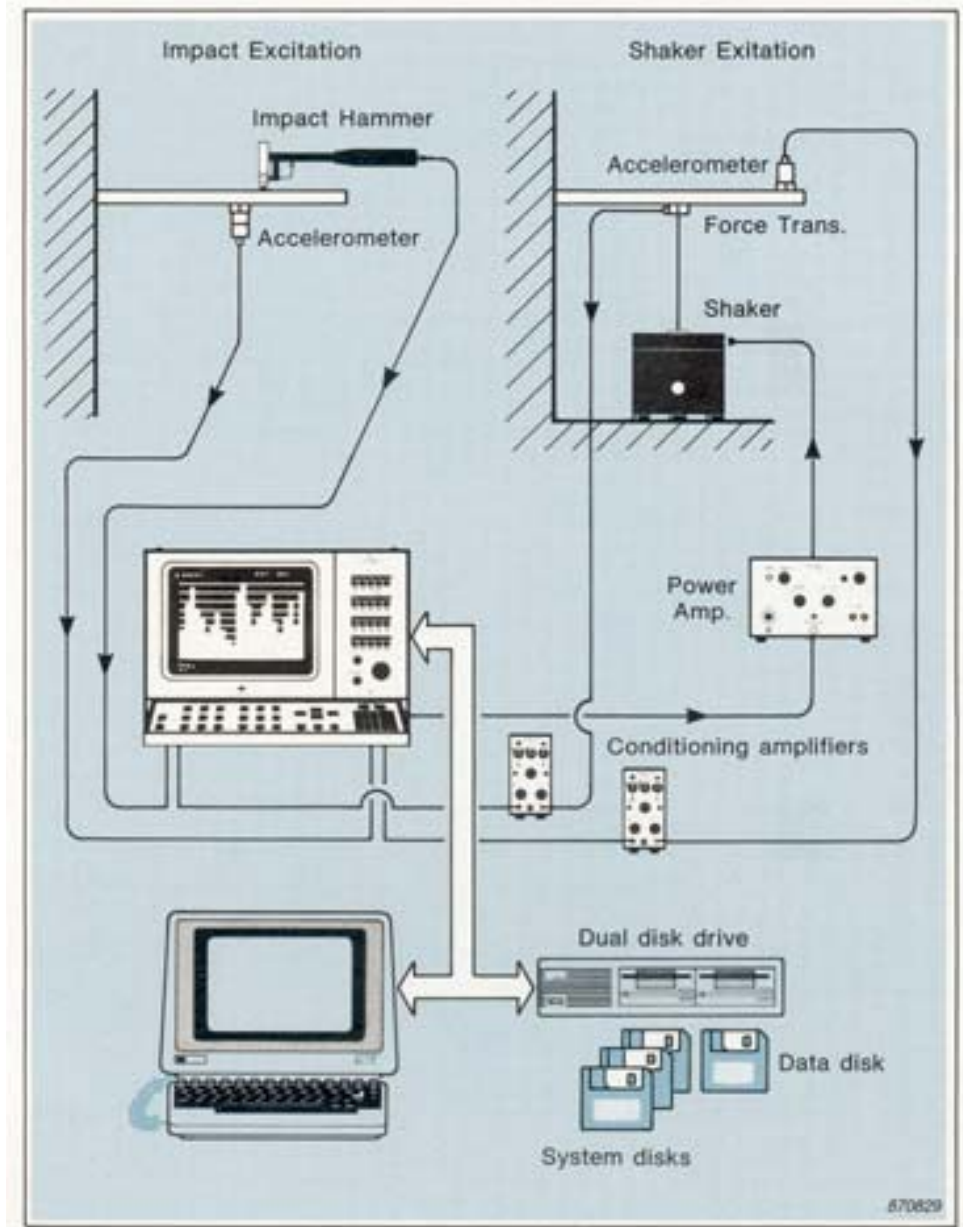
Step 1 - Setting up the Modal Test

- **Choosing the DOFs.** We expect simple mode shapes for the first two elastic vertical modes. So we can limit the number of DOFs to 18, measured in the vertical direction at equally spaced points on the lower frame of the body.

- **Suspension of the test object.** To study the dynamics of the body we try to establish the so-called free-free condition. One method is to suspend the body by elastic cords.

- **Choice of excitation.** To cover the low frequency range, and linearize any non-linear behaviour, we can use random-waveform excitation. The excitation waveform is taken from the signal generator of the analyzer and used, after power amplification, to drive an electrodynamic vibration exciter.

- **Position/connection of exciter and force transducer.** The best position for the exciter, in this case, is at a corner of the body where both the symmetric and asymmetric modes will exhibit maximum motion.

The force transducer is stud-mounted onto the body frame (through a threaded hole) and connected to the exciter by a 4 mm nylon stinger. The exciter is placed directly on the ground where the reaction force will be absorbed.

- **Mounting the response transducer.** As the test structure is heavy and has a smooth surface and the frequency range of interest is low, an accelerometer on a mounting magnet can be used. This mounting technique makes it easy to move the accelerometer between the DOFs.

- **Transducer conditioning and calibration.** The transducer signals are conditioned by charge amplifiers, into which the transducer calibration values have been set. This ensures that the signals supplied to the analyzer are calibrated. A mass-ratio calibration check will then verify the calibration of the total measurement chain.

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Mounting magnet

Accelerometer

Force transducer

Shaker

Power amplifier

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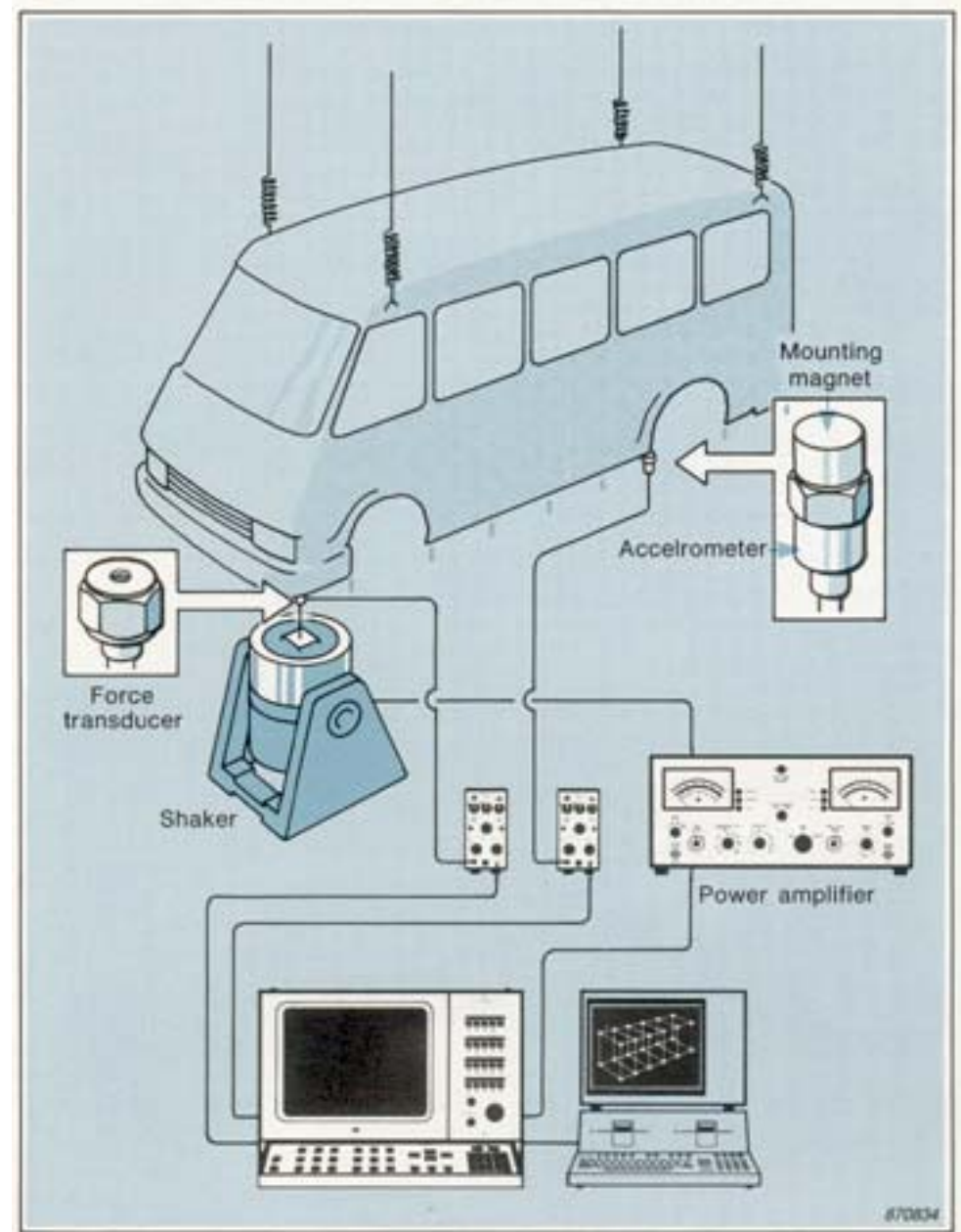
● **Setting up the analyzer.** The optimal measurement parameters for this application are:

Measurement: Dual spectrum averaging mode.

Trigger: Free run - with the trigger disabled, the analyzer will accept data at the maximum rate.

Averaging: Linear averaging - the number of averages is set arbitrarily high. We can stop the process when the estimates become satisfactorily smooth.

Frequency span: 100 Hz.

Center frequency: Baseband - frequency span will start at 0 Hz.

Weighting: Hanning - the optimal window for random excitation.

Ch.A and Ch.B: The input attenuators, filters and calibration values are set in engineering units.

Generator: Random.

● **To check the measurement set-up we can:**

1. Make a driving point measurement, to ensure that:

  • we have a satisfactorily flat force spectrum

  • we have explainable coherence

  • we have positive (accelerance) quadrature peaks

  • we have an anti-resonance between resonances

2. Make a few driving point measurements *at other DOFs,* to check that all the modes of interest are seen at the chosen driving point. For this second check, it is more convenient to use hammer excitation.
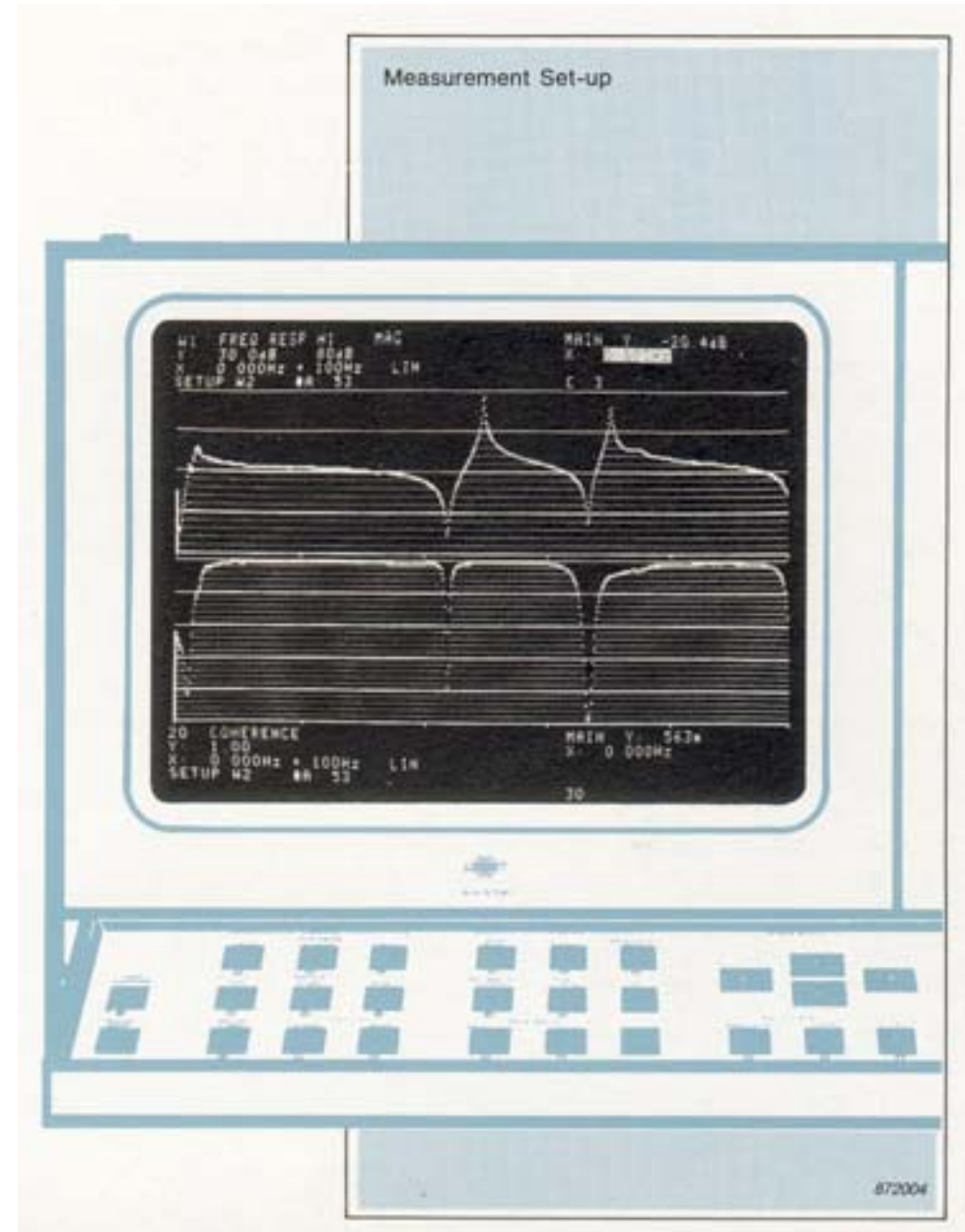
Measurement Set-up
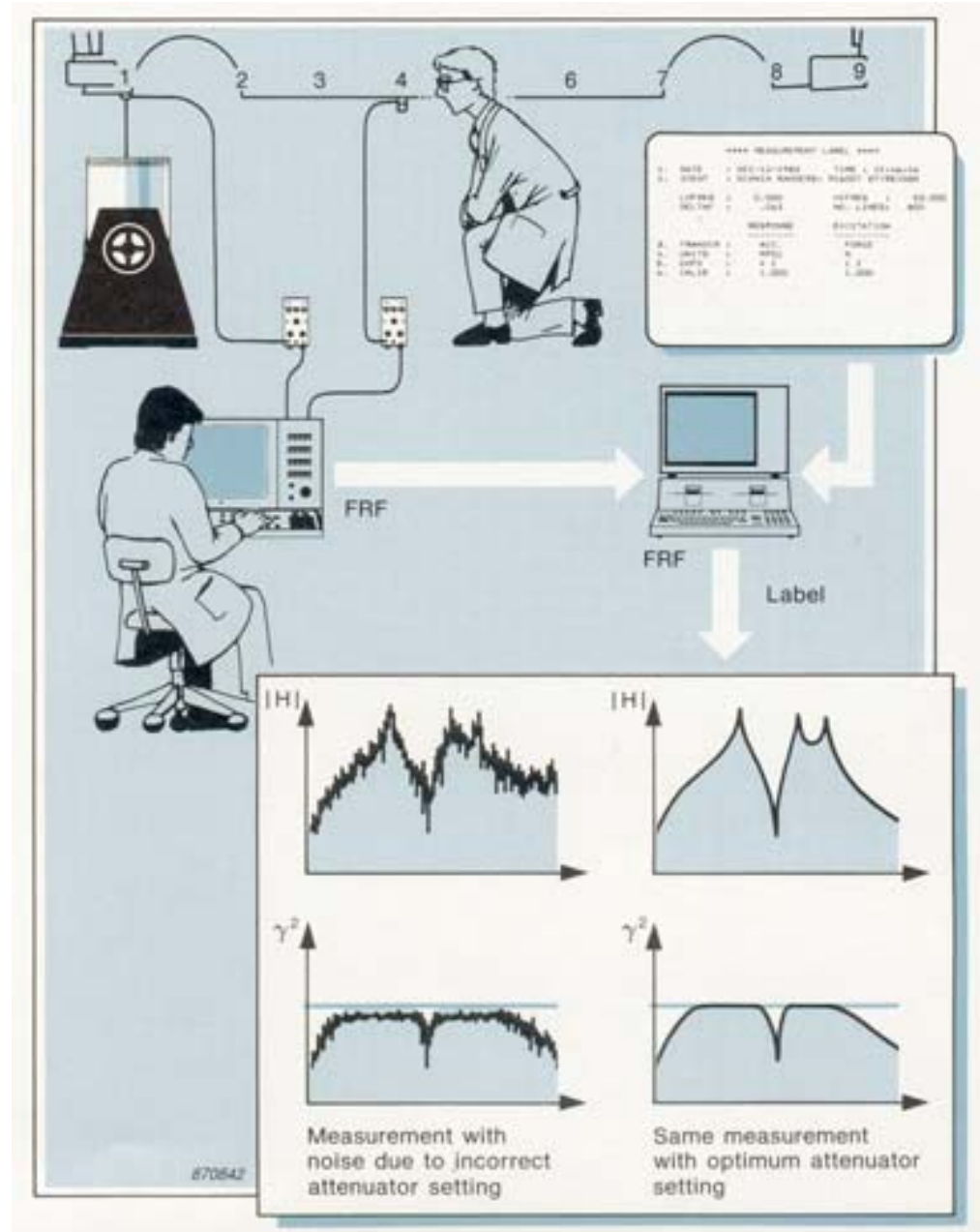
Step 2 - Making the Measurements

In this phase, a set of FRF measurements between the excitation DOF and all the other defined DOFs is made and stored.

The computer helps to manage the data. It transfers the measurements from the analyzer and tells us when each transfer is completed, so that the next transducer location can be selected. The computer also creates the data files, and stores the data on a mass storage medium (disk). The label for each stored measurement records information concerning the DOF measured, the analyzer set-up, calibration, date and time, and any comments we wish to include.

The operator plays an important role by continuously checking the measurements. By watching the coherence and convergence of the FRF on the analyzer display, the operator can decide when to accept the measurements, or when to make corrections.

The program has provision for automation of this process, but this is not recommended:

- The measurement phase is the most critical and important part of the whole operation. The analysis depends primarily on the accuracy obtained at this stage.

FRF

FRF

Label

Measurement with noise due to incorrect attenuator setting

Same measurement with optimum attenuator setting

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Step 3 - Parameter Estimation by Curve-fitting

When the FRF measurements are completed, we can begin the data reduction process to extract the modal parameters. This has three phases:

### 1) Interactive curve-fitting ⟹ frequency and damping

This phase is a manually controlled curve-fit. The operator has to decide which FRF measurement is the best for the application, which modes are of interest, which curve-fitter to use and over what frequency range. During this phase, the global parameters (the *modal frequencies* and *dampings)* are determined. Simultaneously, the computer writes a "recipe", the *autofit* table. This table shows how the operator performed the first curve-fit, and is subsequently used by the computer to fit the rest of the data set.

2) Auto curve-fitting ⟹ modal residues

The computer fits all the measured data in an automatic sequence. The *residues* are estimated and saved in a *fitdata* table.

3) Sort ⟹ mode shapes

In the sort procedure, information contained within the *fitdata* table is converted into (scaled) mode shape data, and saved in the *shape* table. The computer also transforms the data from local to global coordinates, and adds motion to non-measured DOFs expressed as linear combinations of measured DOF motions.

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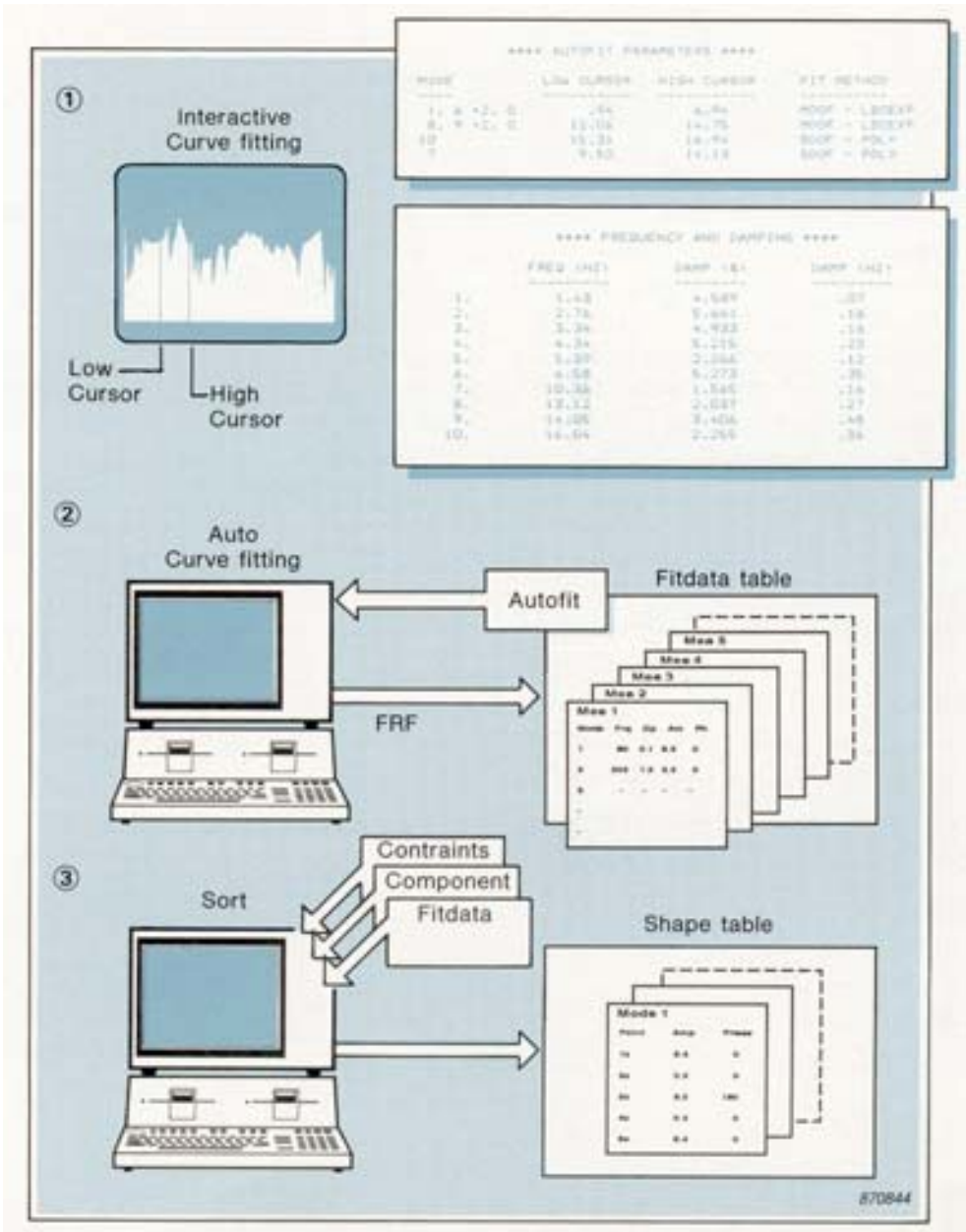
Step 4 - Documentation of the Test

Documentation can be in the form of a printed result table. But for large data sets graphic presentations are much more powerful.

• The geometrical model

From the definition of the modal model, no geometrical information (apart from that of the defined DOFs) is included in the dynamic description. For documentation purposes, we need to include some geometry so that the pictures produced will resemble the test object.

In this test we measured only vertical DOFs in one plane (the body frame). This relatively simple geometric structure can be described in a rectangular coordinate system (by updating the *coordinate* table). To produce a picture, we specify how to draw lines between the coordinates by updating the *display sequence* table.

• Animation

Mode shapes can be defined as a sampled set of relative deflections over the structure, represented by the mode shape vector. The principle of animation may be visualized as a cartoon strip showing the mode shape drawn with a harmonically varying scale, and displayed in a continuous sequence on the computer display. Optional viewing facilities provide; rotation around three axes, zoom and panorama, variable animation amplitude and speed, an overlay of different modes or the undeformed geometry. Animation is a very informative presentation for mode shapes.

• Mode shape hard-copy

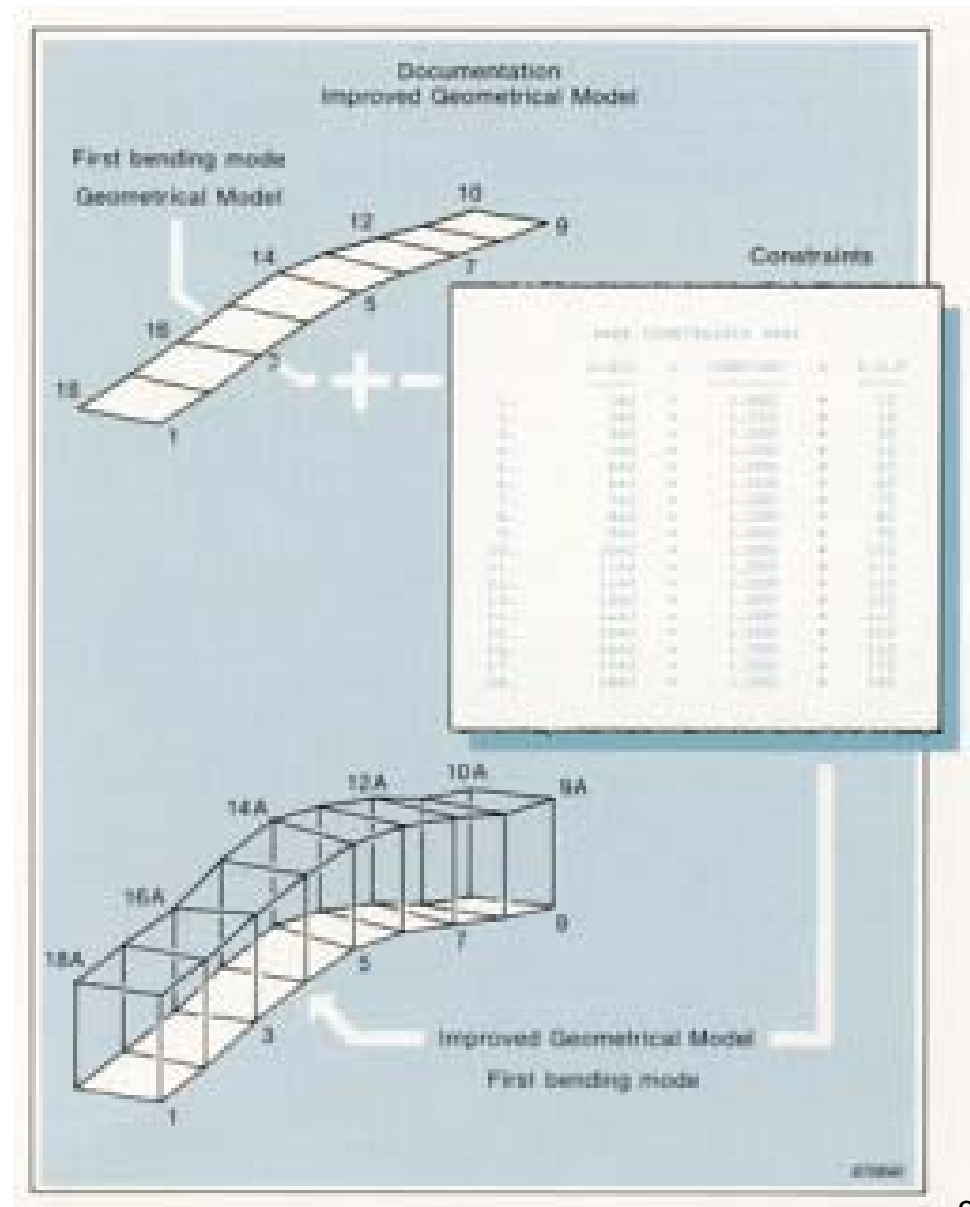A digital plotter can be used to make hard-copy plots of the mode shapes.
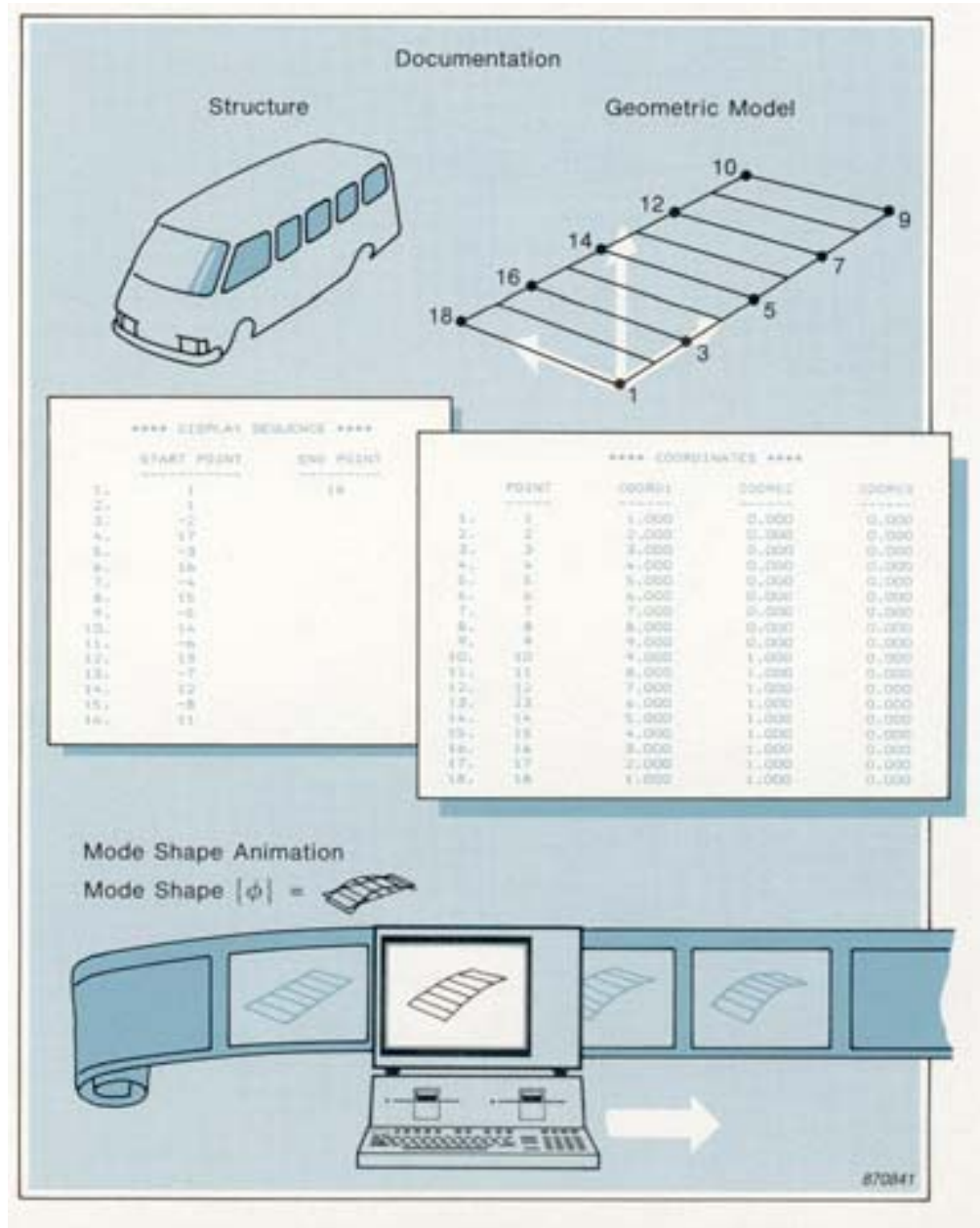
• Improved geometrical model

Our simple geometric model of the lower body frame can be extended to include coordinates on the minibus roof. The resulting display will then be a box structure giving a more realistic representation of the minibus body.

There is a provision in the software for including unmeasured DOFs in a *constraints-equation* table. The constraint is that the motion of an unmeasured DOF must be a constant multiplied by the motion of a measured DOF.

In our example we may assume that, for the two modes of interest, the displacements of the roof and the lower frame are identical. We can then enter the unmeasured DOFs (1AZ to 18AZ), with constant values of 1,00, into the constraints table and modify the *display-sequence* table.

The computer will add motions to the unmeasured DOFs during the sort procedure, and display the new geometrical model illustrated.

Documentation
Improved Geometrical Model

First bending mode
Geometrical Model

Constraints

Improved Geometrical Model
First bending mode

The Dynamic Modal Model

• The ultimate result of the modal test

The direct result of a modal test is the visualized mode shapes and the associated resonances. But beyond that, we have an instrument with which we can make a complete dynamic mathematical model of the test object.

• What is a dynamic model?

A dynamic model is a mathematical formulation of the dynamic properties of the structure in a discrete set of *points* and *directions.* It is *not a model of the physical structure.* For example, if a structure had only one input and one output, a model could be:

$$X(\omega) = H(\omega) \cdot F(\omega)$$
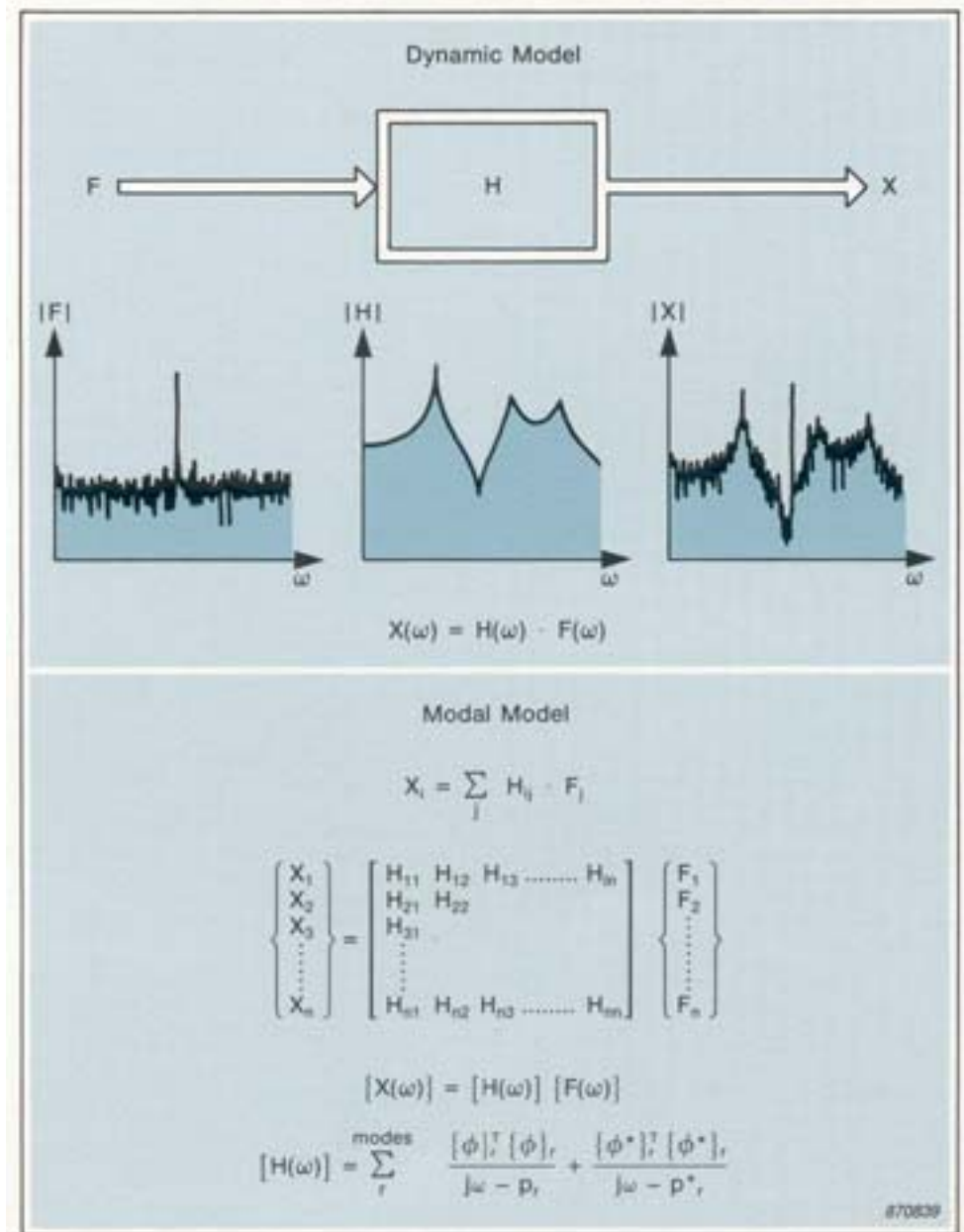
where the parameter in the model is an FRF.

a What is a modal model?

A modal model is a generalized dynamic model:

$$\{X\}=[H] \cdot \{F\}$$

where the vector {X} is a list of the vibration spectra in the specified DOFs. {F} is a list of the excitation spectra for the same DOFs, and [H] is a matrix of all the possible input/output FRF combinations. This is called a modal model because [H] can be calculated from the estimated modal parameters.

The advantage of this formulation is that the *parameters are measurable.* Although we only measured one row, or one column, of [H], we can calculate any other element from a knowledge of the mode shapes, the modal frequencies and the dampings.

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Dynamic Model

$$X(\omega) = H(\omega) \cdot F(\omega)$$

Modal Model

$$X_i = \sum_j H_{ij} \cdot F_j$$

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & \cdots & H_{1n} \\ H_{21} & H_{22} & & & \\ H_{31} & & & & \\ \vdots & & & & \\ H_{n1} & H_{n2} & H_{n3} & \cdots & H_{nn} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}$$

$$[X(\omega)] = [H(\omega)] \, [F(\omega)]$$

$$[H(\omega)] = \sum_r^{modes} \frac{\{\phi\}_r^T \{\phi\}_r}{j\omega - p_r} + \frac{\{\phi^*\}_r^T \{\phi^*\}_r}{j\omega - p^*_r}$$

Checking and Applying the Model

Because it is relatively easy to compare shapes and frequencies, qualitative verification of analytical solutions can easily be made from the estimated modal parameters. The quantitative potential of the model, however, is also worth considering.

• Adequacy of the model

For the modal model to be useful in a quantitative sense, it must be a sufficiently precise and complete representation of the structure.

• Checking precision by FRF synthesis

If the modal test and calibration checks have been made correctly, then our modal data will give an accurate description of the dynamic properties. A simple procedure can be used to test the accuracy.

During the test, we measured either one complete row (impact test), or one complete column (attached exciter) of the FRF matrix. From this measured data, the computer program can be used to synthesize a non-measured FRF. If we then measure the corresponding FRF on the test object, and compare the measurement with the prediction, we can see how accurate and adequate our modal model is.

Accuracy can be observed around the modal frequencies, where the peaks should fit exactly. The adequacy must be judged by the operator on the basis of how well the two functions compare between modes, and in relation to the future application of the model.

$$|H_{ij}| = \sum_{r=1}^{m} \frac{\phi_{ir}\phi_{jr}}{j\omega - p_r} + \frac{\phi^*_{ir}\phi^*_{jr}}{j\omega - p_r^*}$$
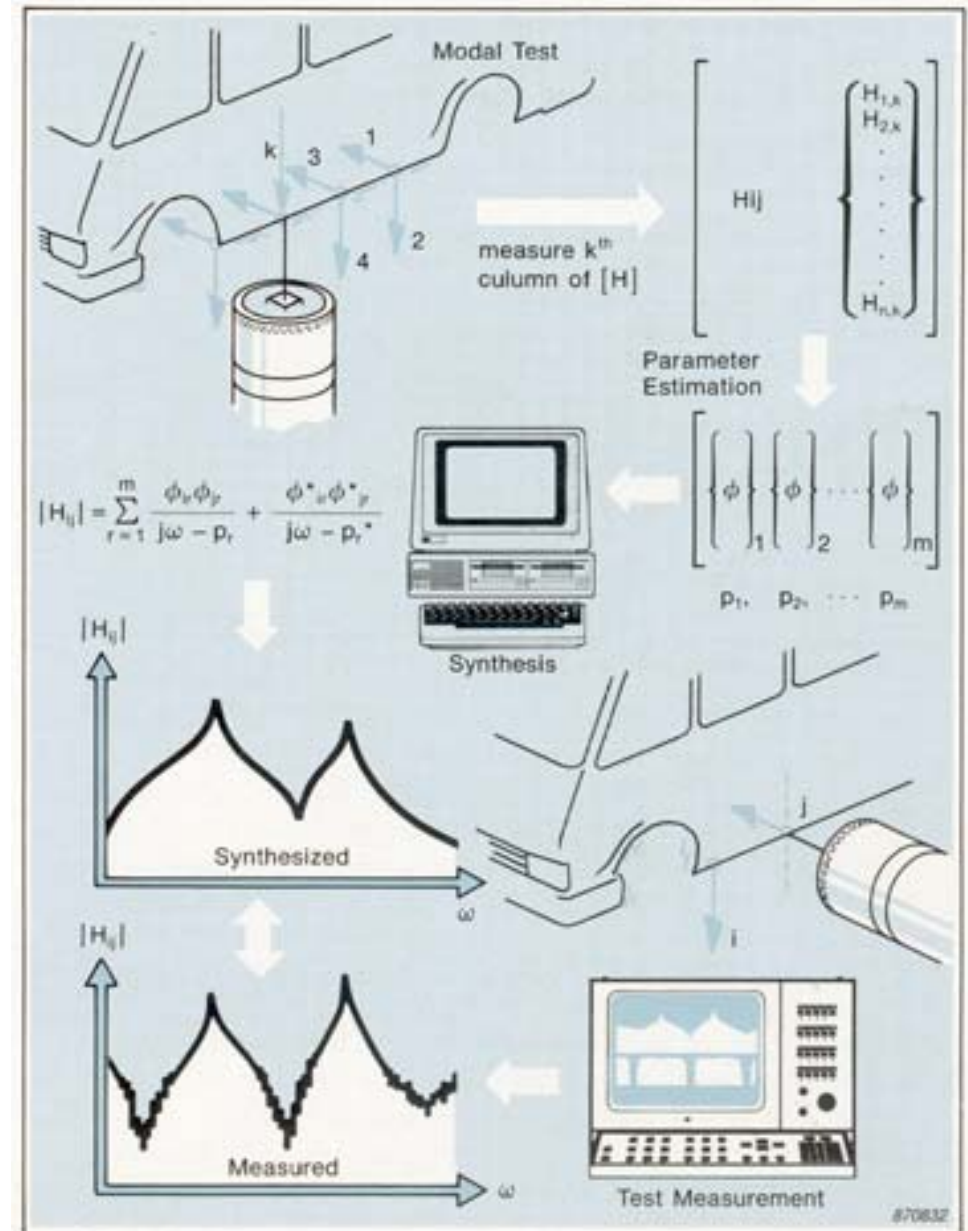
Considerations of Model Completeness

From a theoretical point of view, the modal model formulation is exact since no approximations are introduced. But how good is the measured/estimated model? Assuming that our initial assumptions of linearity, etc., hold true, then there is a potential pitfall due to *truncation.*
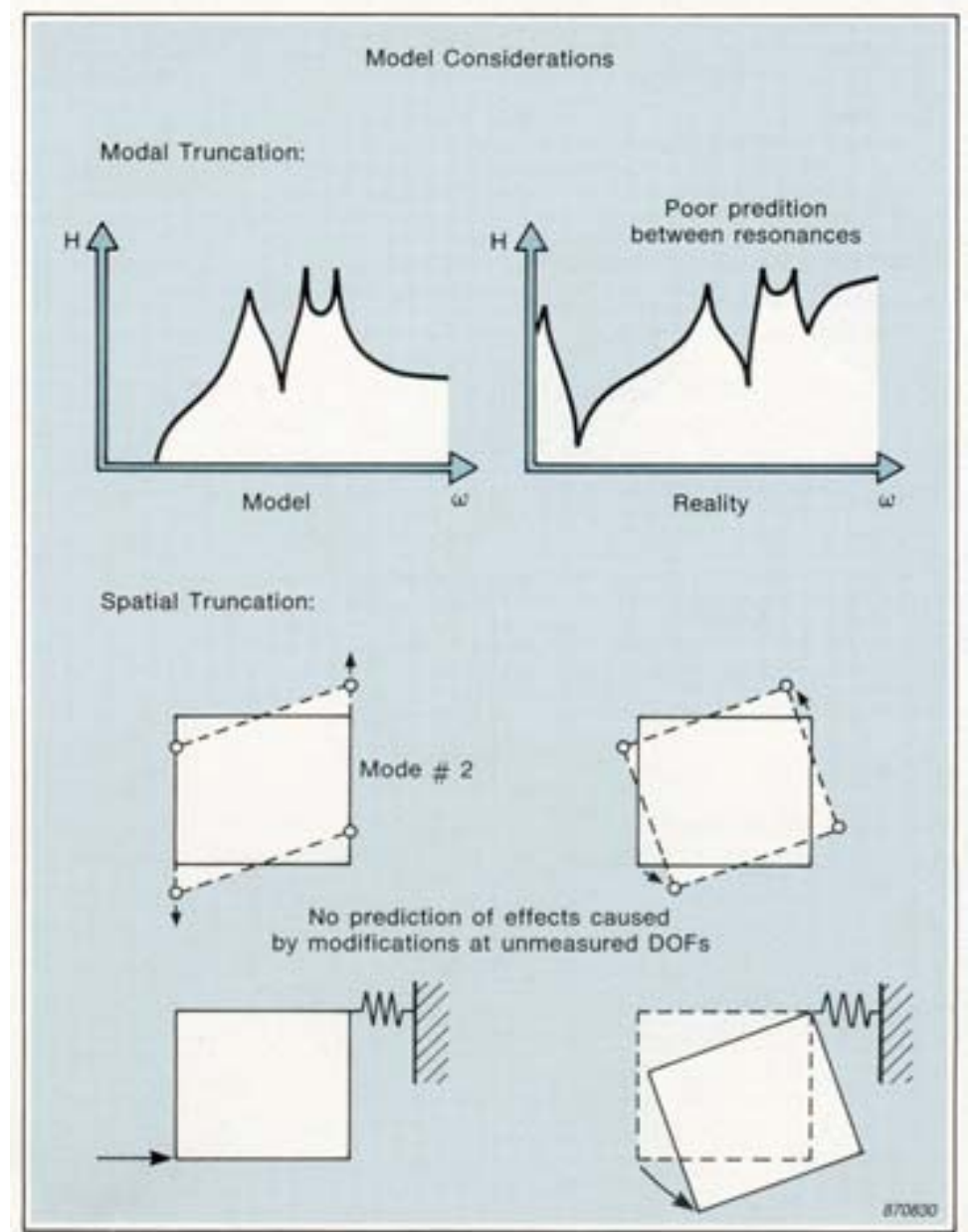
• Mode truncation

As we have to limit the frequency range of a test, not all the modes of the structure are included. In practice, we often ignore the rigid body modes, at the very low frequencies, and also modes present only at local parts of the structure. We also try to keep the frequency range as low as possible to obtain maximum frequency resolution. This means that we truncate the frequency domain description, which can lead to reduced accuracy of the model particularly between the resonances.

• Spatial truncation

We work with a finite discrete set of DOFs to describe a continuous structure. But each point on the structure can theoretically move in six directions. The lack of measurements in some of these directions, and the finite number of measurement points used, represent a spatial truncation.

Let us look again at our modal test. The minibus body was described on the basis of a set of vertical measurements. From the results obtained, the second rigid body mode (torsion) looks more like shear since no information on horizontal motion was measured.

A model cannot be used to predict the effect of forces or modifications at points/directions where no measurements were made.

Model Considerations

Modal Truncation:

Poor predition between resonances

Model ω Reality ω

Spatial Truncation:

Mode # 2

No prediction of effects caused by modifications at unmeasured DOFs

Computer Simulations: What if?

Computer simulations are the advanced applications of the precise and complete modal model. Simulations can help us to answer many of the "What if?" questions applicable to optimizing prototype design, problem-solving, and studying the structure under estimated operational conditions.

Response simulation can predict the vibration response of the structure if different excitation forces are applied at any of the defined DOFs.

Modification simulation can be used to predict what will happen to the model in terms of its modal parameters if physical modifications (mass, stiffness, damping and sub-structures) are applied.

Verification. The predicted response can be converted to noise, strain, fatigue etc., for comparison with reference data, design criteria or standards. If the results are not satisfactory, the engineer will then have to formulate a remedy.
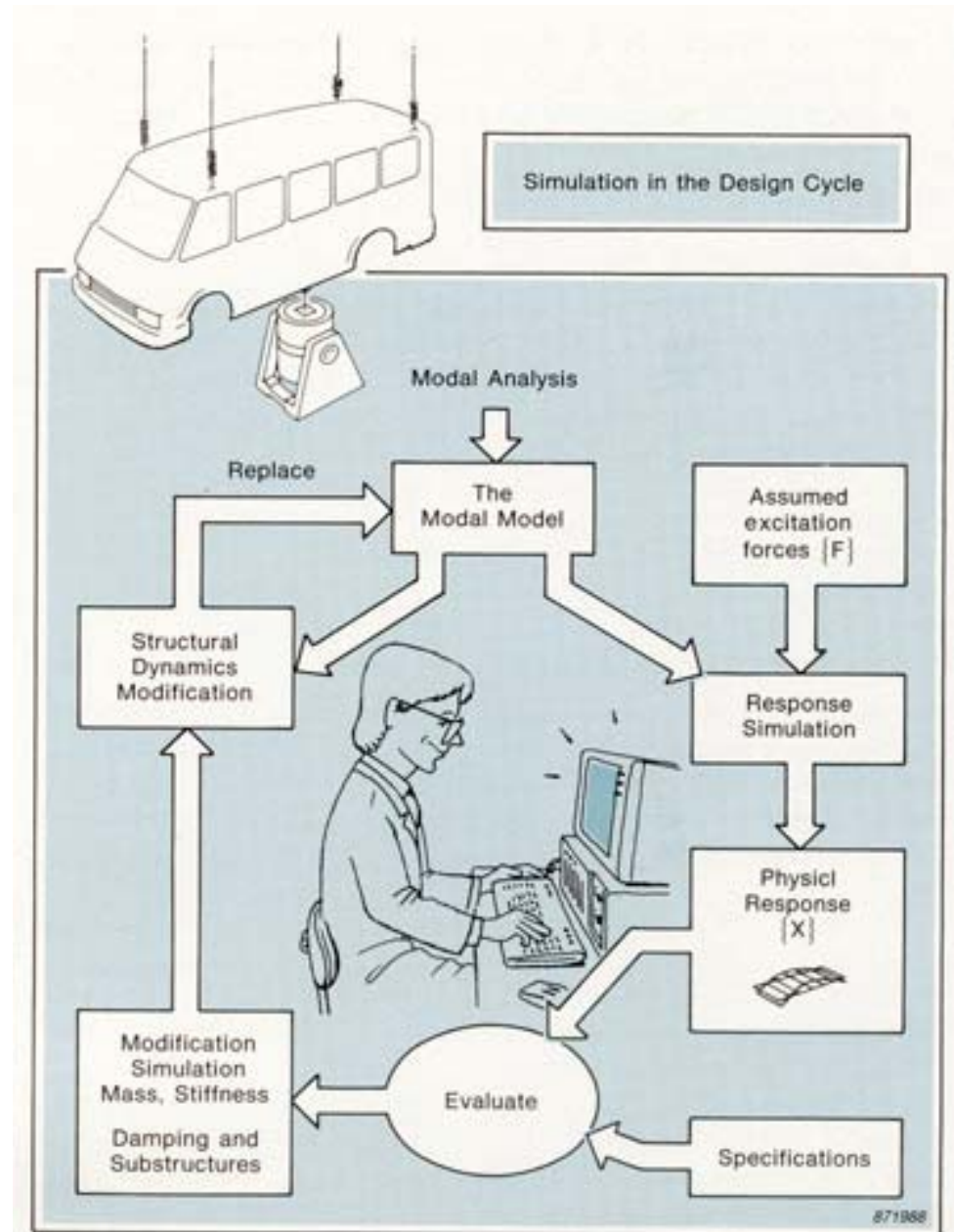
Simulation in the design cycle

The cyclical process of:

Response simulation ●

● Verification

Modification simulation ●

can be repeated as many times as necessary.

Due to the high speed of computers and software the time taken for a complete cycle is only a few minutes, a true optimization is therefore possible in a very short time.

Simulation in the Design Cycle

Modal Analysis

Replace

The Modal Model

Assumed excitation forces [F]

Structural Dynamics Modification

Response Simulation

Modification Simulation Mass, Stiffness Damping and Substructures

Physicl Response [X]

Evaluate

Specifications

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Response Simulation

To predict a physical vibration response, we need to load the modal model with a specific set of simulated physical forces.

● **Excitation for simulations**

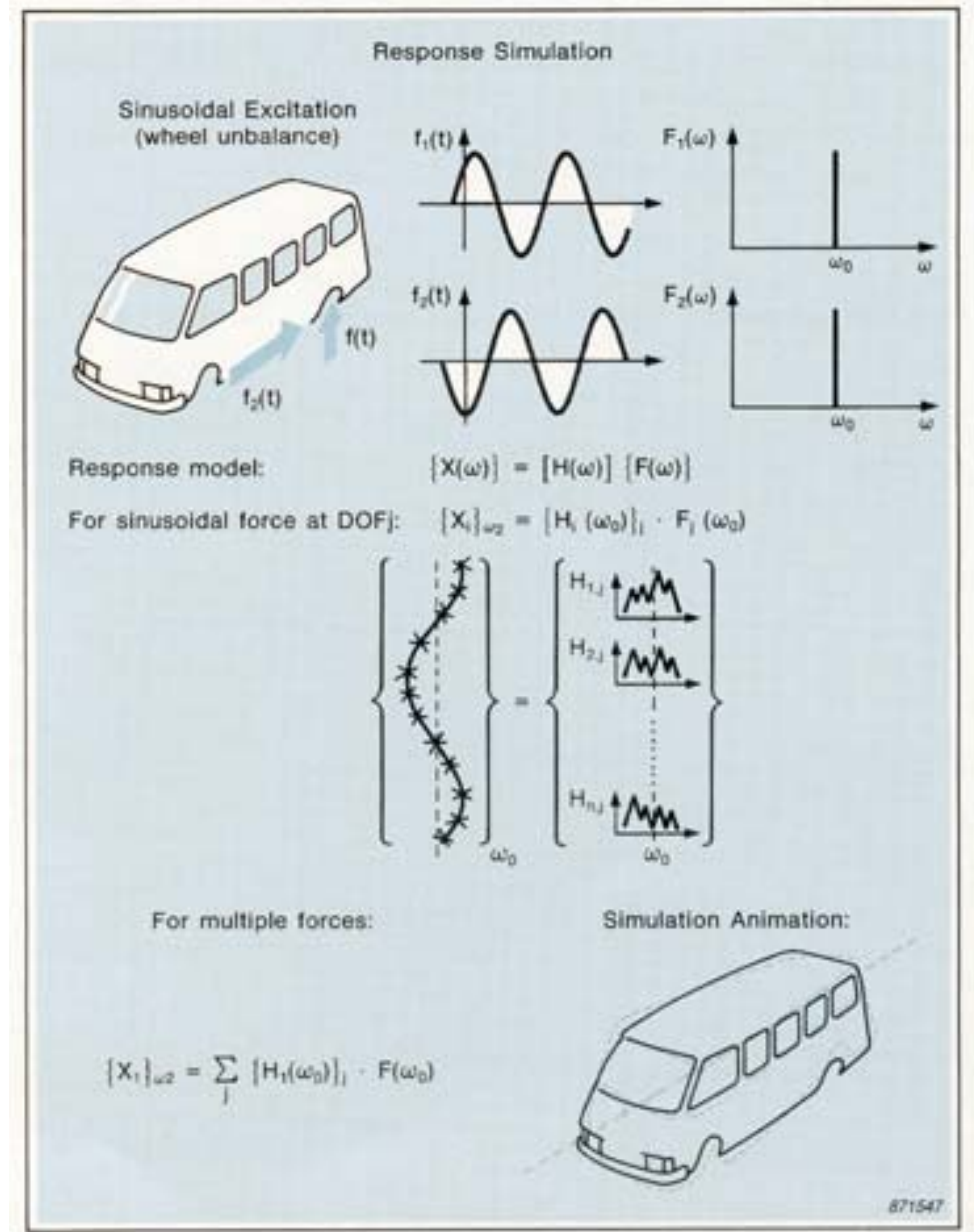Two types of excitation, sinusoidal and broadband, can be used. They require two different kinds of simulation.

• **Sinusoidal excitation**

Sinusoidal excitation is realistic for many applications. In operational environments, structures are often exposed to the sinusoidal excitation of free forces, or to moments produced by rotating components. Implementation is simple. We need a list of the physical deflections $\{x\}_{\omega 0}$ (the Operational Deflection Shape) due to excitation at one DOF. With only one frequency present, there is just one line in the response spectrum - one value per DOF.

If more than one excitation force is required, the response will be the sum of the individual responses. A rotating unbalance, for example, can be modelled as two orthogonal forces, 90° out of phase.

Sinusoidal animation

Animation is achieved by calculating the deformation vector, using the the same basic routine as for the mode shape animation. If more than one frequency is used, true animation is not feasible, but a motion pattern representing the vibration envelope can be shown.


Response Simulation

Sinusoidal Excitation (wheel unbalance)

$f_1(t)$ $F_1(\omega)$

$f_2(t)$ $F_2(\omega)$

$f(t)$

$f_2(t)$

Response model: $\{X(\omega)\} = [H(\omega)]\,\{F(\omega)\}$

For sinusoidal force at DOFj: $\{X_i\}_{\omega 2} = \{H_i(\omega_0)\}_j \cdot F_j(\omega_0)$

For multiple forces:

$\{X_i\}_{\omega 2} = \sum_j \{H_i(\omega_0)\}_j \cdot F(\omega_0)$

Simulation Animation:

• Broadband excitation

Broadband simulation is used for random forcing inputs. This could be required, for example, to predict comfort in a car driven over a specific (standard) surface, or to predict component fatigue in a turbulent environment.

The Force Autospectrum G_{FF} can be obtained from calculations, from standards, or from measurements. The software can then synthesize FRFs between the force input DOF and any other DOF in the modal model.

With this type of excitation, we are dealing with statistical parameters (rather than discrete values), and the results have to be evaluated by using statistical inference.

Response Simulation

$$G_{XX_i} = \sum_i |H_{ij}|^2 G_{FF_i}$$
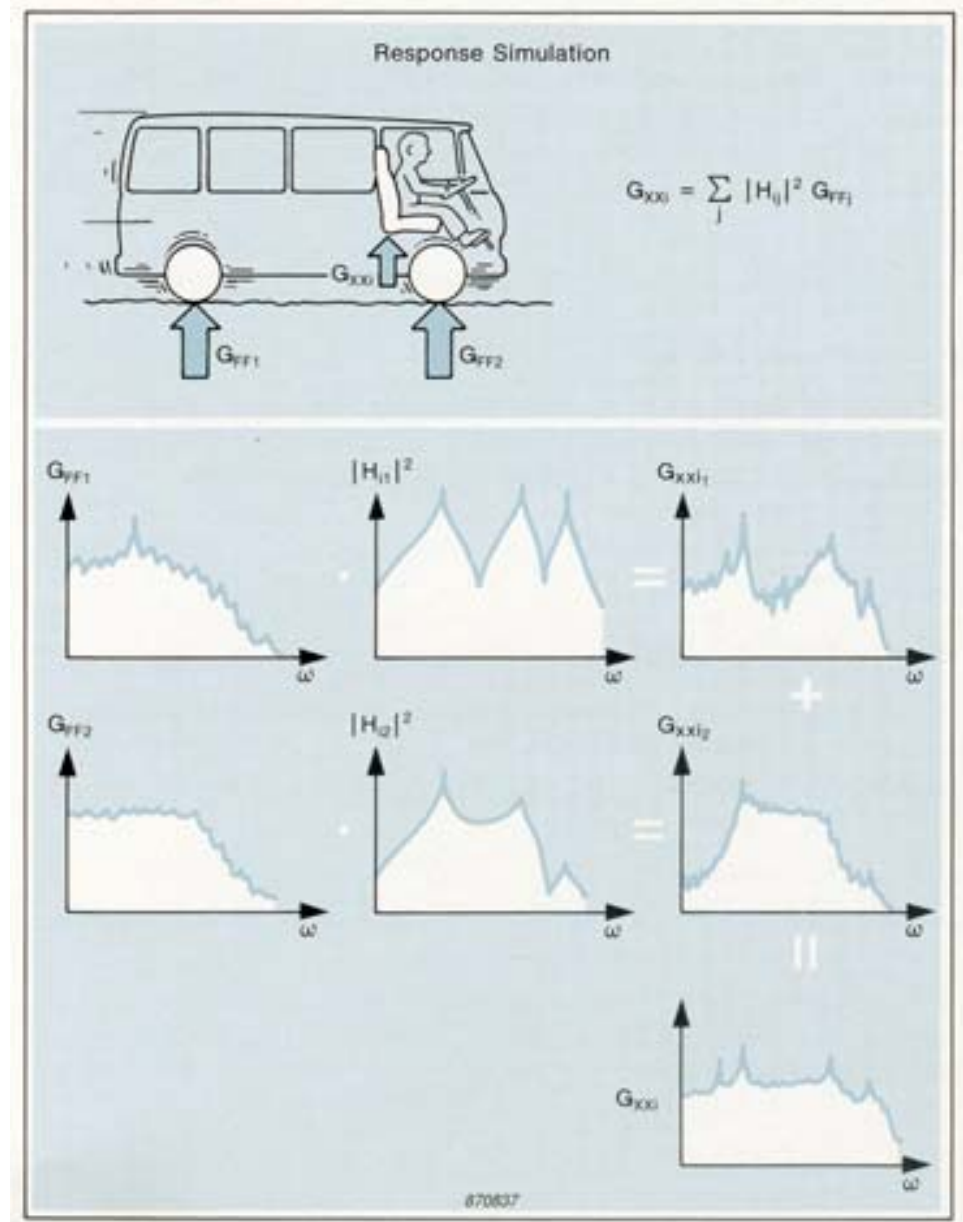
Modification Simulation

● **Problem-solving: "What if?"**

Many noise and vibration problems found in either the design phase, or the operation, of a structure are caused by resonances. Resonances can cause mechanical amplification of normal operational forces, resulting in an unacceptable structural response.

In such cases, the engineering task is to suggest a solution. This will often be a structural modification (in mass, stiffness or damping) aimed at shifting the resonance. One path to a solution is through trial and error, but this tends to be expensive in time, material costs and quality.

With a modal model available, alternative modifications can be evaluated using a computer. The "What if?" approach can be made many times before the hardware modifications are implemented and finally tested.

● **Operational modifications: "What, When?"**

A similar problem exists when the actual operation of the structure influences its dynamics.

"What" happens "When" the structure:

  • takes payload (a mass modification)?

  • connects to ground (a stiffness modification)?

  • connects to another structure (sub-structuring)?

By using a modal model, the effect of these events can be predicted by computer simulation. The process of predicting the modified dynamic properties produces a new modal model - the very instrument required to predict the new vibration response.
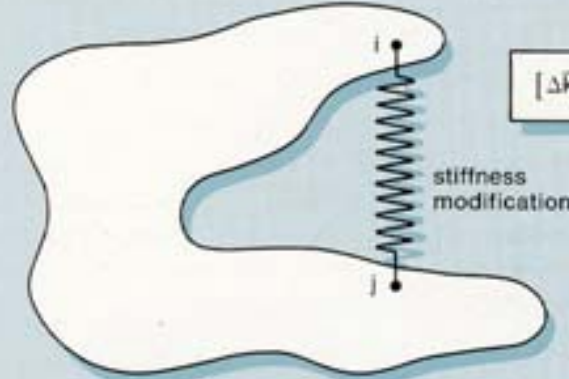
Modification Simulation
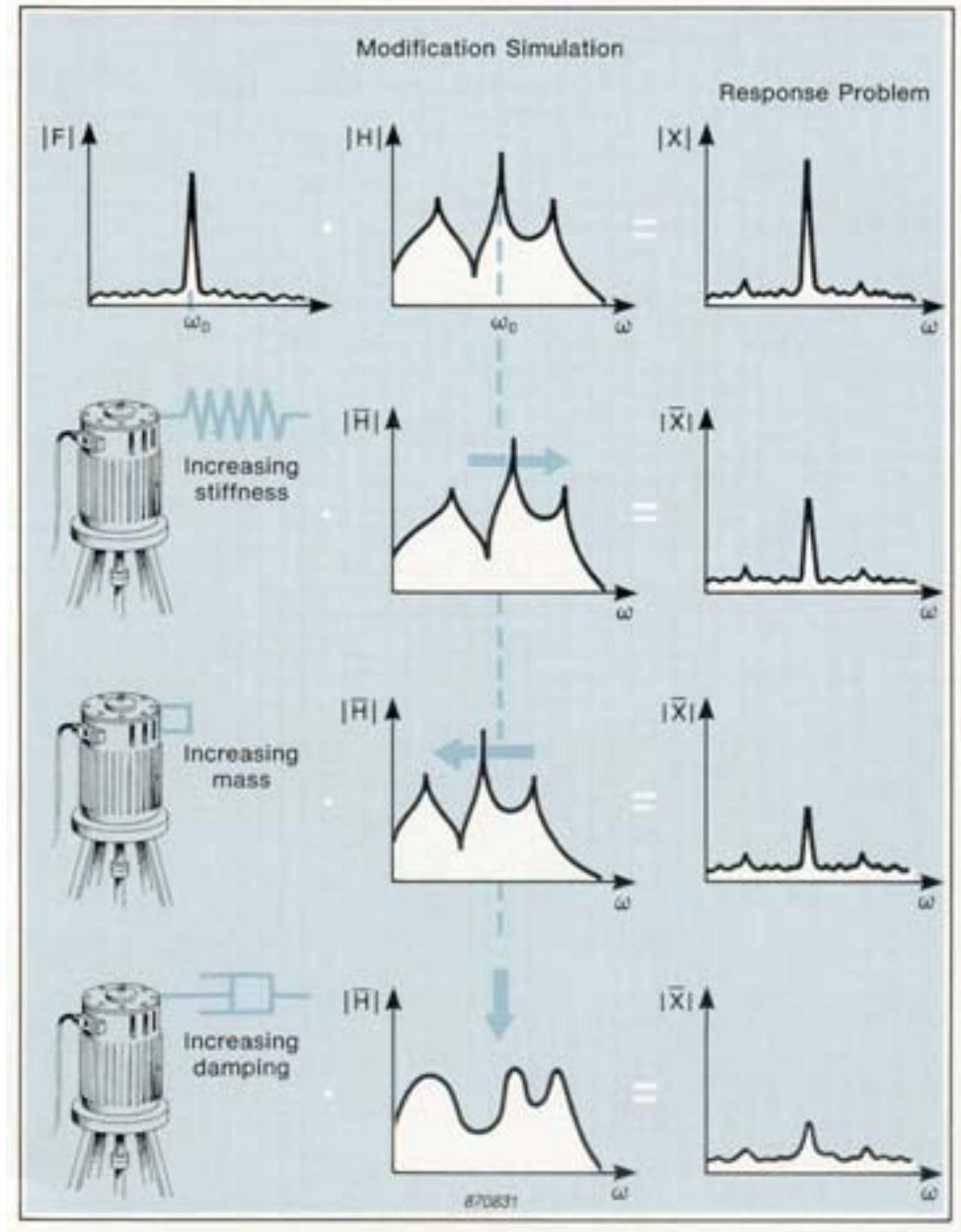
Applying Modifications

• Modification calculations

We do not generally know the spatial parameters for the test structure, only the parameters in the modal coordinates. A physical modification, described locally in spatial parameters, will spread its effect into all the modal coordinates through the modal transform.

The equation for the modified structure is a standard eigenvalue problem, the solution of which gives the new modal frequencies and dampings, and the new mode shapes.

Since all calculations are based on the modal model, *modifications can only be simulated at measurement points, and in measured directions.* No approximations are used so the method is exact. Its accuracy depends entirely on the quality of the measured model.

Physical Modifications in Modal Coordinates

$$[\Delta\bar{k}] = [\phi]^T [\Delta k] [\phi]$$

stiffness modification

Solving: characteristic equation

$$\left[[\backslash 1 \backslash] S^2 + [2\alpha] S + [\phi]^T [\Delta k] [\phi] + [\backslash \omega_s^{\omega} \backslash] \right] \{q(s)\} = 0$$

yields: $p_1, p_2, \dots p_m$ = the new pole locations

and $\{\phi\}_1, \{\phi\}, \dots \{\phi\}_m$ = new mode shapes

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Implementing Modifications

● **The building blocks** in a modification program are:

• Point masses

• Massless, linear springs

• Viscous dampers

In addition, some compound elements may be available:

 • Tuned absorbers

 • Beam elements (rib stiffners)

 • Sub-structures

● **Software facilities**

Modification software-packages provide the user with a universal tool which can be used in a variety of ways. The direct application has already been discussed, but it is also possible to work from the other direction. We can enter a set of desired modal frequencies and let computer calculate the modifications required.

● **Modelling physical modifications**

Practical modifications cannot be simulated as a point mass or a massless spring, but they have to be modelled by combining these basic elements.

A rod welded between two DOFs in the structure might be modelled by one stiffener in the main axis. If lateral motion is present, another two stiffeners, perpendicular to each other in the horizontal plane, will improve the model. As physical rods are not massless, the model can be further improved by making two mass modifications, with 1/3 of the physical mass of the rod at each of the two DOFs.

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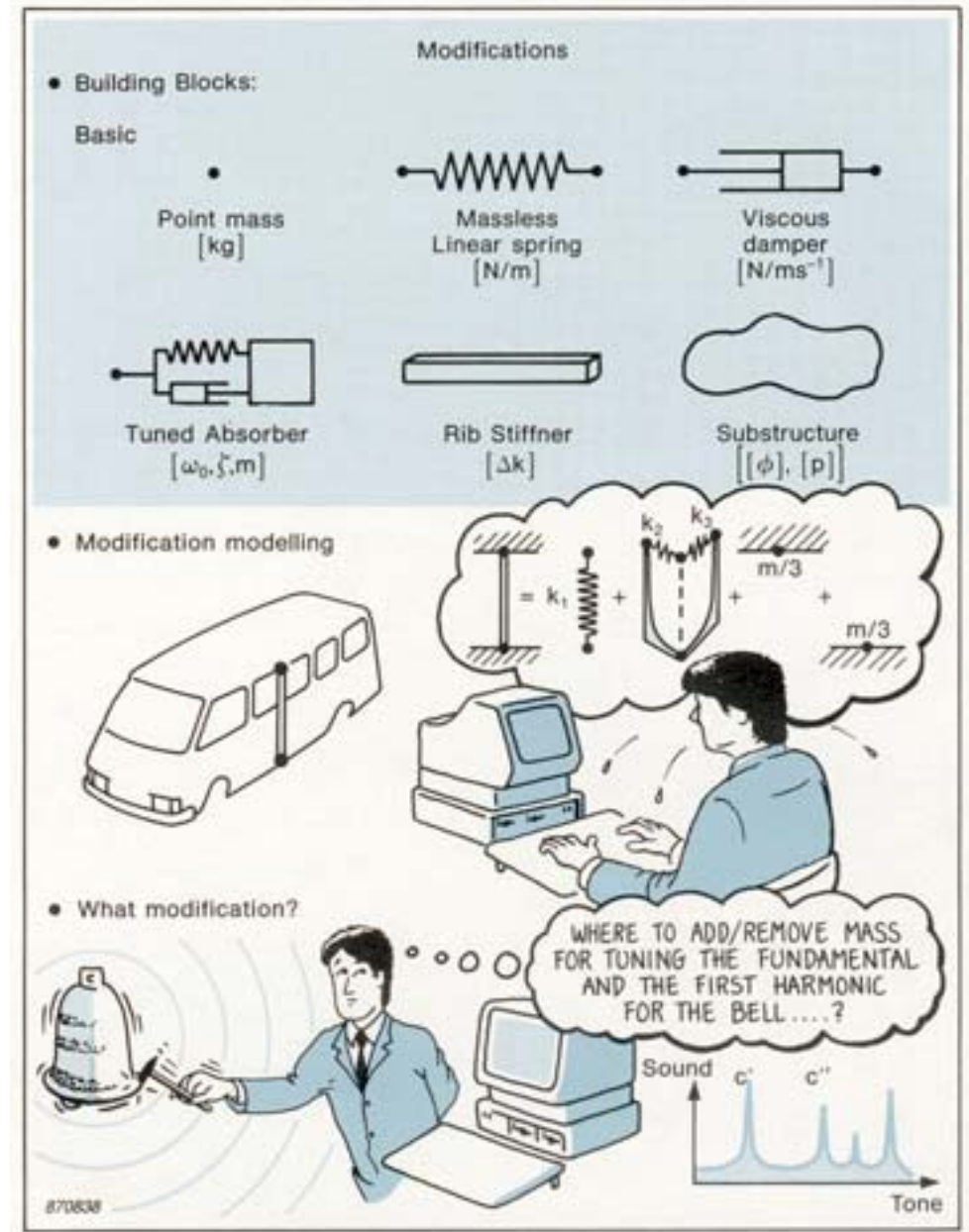
Case Story: An Application of Synthesized FRFs

● **A ship vibration problem**

Under normal operating conditions, severe instrument reading problems were experienced on the bridge of the ship. The immediate cause of the problem was excessive vibrations at the bridge.

An analysis of the operational vibration signal indicated that the response increased exponentially with propeller speed. The vibration spectrum showed a concentration of energy at the propeller blade-passing frequency, which identified the propeller as the primary source of the vibration.

The problem then was to determine where the energy was entering the structure. The potential sources were; the stern bearing (two degrees of freedom), the thrust-bearing (one degree of freedom), the gearbox (two degree of freedom), or pressure fluctuations integrated over the hull in the propeller vicinity.

● **The investigation**

It was decided to base the investigation on a modal test with a limited number of DOFs. Excitation was implemented by an eccentric mass exciter, welded to the stern deck. The exciter was run in fast frequency sweeps yielding a relatively flat energy spectral density while the FRF measurements were made.

Case: Diagnosis of Ship Vibrations

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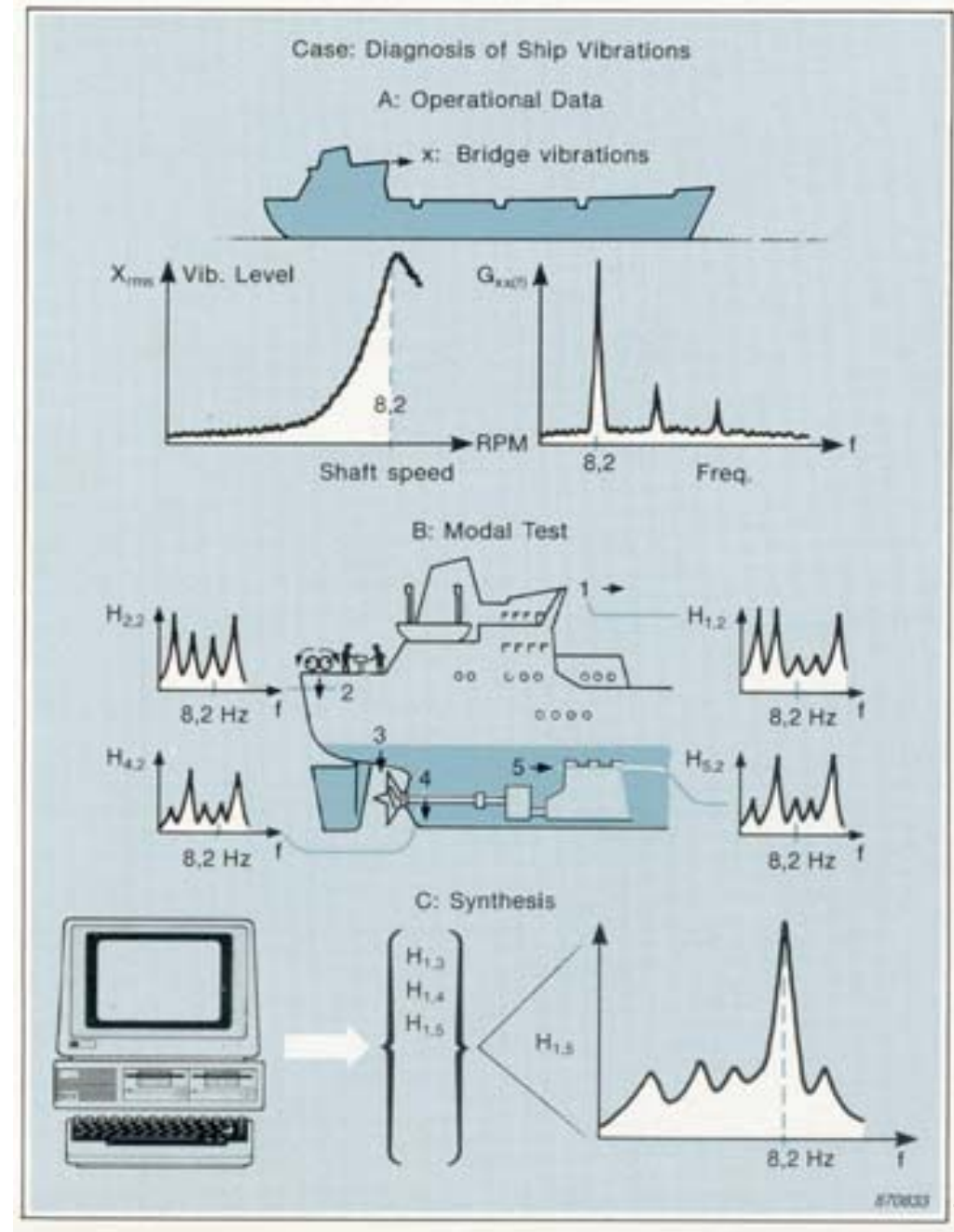
● The results

None of the measured FRFs showed strong resonances at the problem frequency (8,2 Hz), not even when the response was measured at the bridge. But the operational forces could not have been entering the structure at the upper deck level, therefore on the basis of the estimated modal parameters, the FRFs between the bridge and all of the potential input DOFs were synthesized.

The problem was identified when the synthesized FRF between the thrust bearing and the bridge was found to exhibit an extremely high mobility at 8,2 Hz.

● The solution

A stiffening modification to the thrust-bearing structure reduced the mobility by a factor of five. Subsequent measurements demonstrated that the operational vibration level had been reduced by almost the same factor.

Further Reading

D. J. EWINS
  "Modal Testing: Theory and Practice" *Research Studies Press Ltd., Letchworth, Herts, England.*

K. ZAVERI
  "Modal Analysis of Large Structures - Multiple Exciter Systems" *Brüel&Kjær* BT 0001-12

IMAC Conference Papers
  "Proceedings of the International Modal Analysis Conference" *Union College, Schenectady, N.Y. 12308*

SEM Journal
  "The International Journal of Analytical and Experimental Modal Analysis" *The Society for Experimental Mechanics, Inc., School Street, Bethel, CT 06801*

Symbols and Notation

SIGNAL ANALYSIS

Time Domain

T   Time
τ Time delay
f(t)   Force signal
x(t)   Displacement signal
$\dot{x}(t)$ **Velocity signal**
$\ddot{x}(t)$ **Acceleration signal**

Frequency Domain

f $= \omega/2\pi$ Frequency (Hz)
ω $= 2\pi f$ Angular velocity or frequency (rad/sec)
F(f)   Force Spectrum
X(f)   Displacement Spectrum

SYSTEM ANALYSIS

Time Domain

$h(\tau)$ Impulse response function

Frequency Domain

$\gamma^2(\omega)$ Coherence function estimate
$H(\omega)$ Frequency response function of system
$H_1(\omega)$ Frequency response function estimate
$H_2(\omega)$ Frequency response function estimate

General notation

$\mathcal{F}$ Fourier transform
$\mathcal{H}$ Hilbert transform
$\mathcal{L}$ Laplace transform

PHYSICAL PARAMETERS

m   Mass
k   Stiffness
c   Damping coefficient
c_c Critical damping coefficient
ω Angular velocity

MODAL PARAMETERS

i   Response DOF
j   Excitation DOF
m   Number of modes
r   Index for the rth mode
ω_{0r} Undamped natural frequency for r^{th} mode
ω_{dr} Damped natural frequency for r^{th} mode
ζ_r Damping ratio for r^{th} mode
ϕ Modal displacement
$\{\phi\}_r$ Scaled mode shape vector for r^{th} mode
$\{\Psi\}_r$ Unsealed mode shape vector for r^{th} mode
$q_r(t)$ Modal coordinate for r^{th} mode as a function of time
$Q_r(f)$ Modal coordinate for r^{th} mode as a function of frequency
$\Gamma_r(t)$ Generalized modal force in time domain
$\Gamma_r(f)$ Generalized modal force in frequency domain
σ_r Decay rate for r^{th} mode
p_r Pole location for r^{th} mode
a_r Scaling factor for r^{th} mode
R_{ijr} Residue r^{th} mode between DOF i and j

MATHEMATICAL SYMBOLS

∡	Angle or Phase angle
Σ	Sum
( )*	Denotes the complex conjugate
j	Indicates an imaginary entity
(^)	Indicates an estimate
#	Number
Im[ ]	Imaginary part
Re[ ]	Real part
| |	Magnitude
≈	Approximately equal to
∝	Proportional to
*	Indicates convolution
—	Modified function or parameter
x, y, z	Coordinate axes

UNITS

N	Newtons
m	Metres
s	Seconds
kg	Kilograms
Pa	Pascals

ABBREVIATIONS

DOF	Degree of Freedom
SDOF	Single Degree of Freedom
MDOF	Multiple Degree of Freedom
SDM	Structural Dynamics Modification
FRS	Forced Response Simulation
TA	Tuned absorber
FRF	Frequency Response Function

MATRIX NOTATION

[m]	Mass matrix
[k]	Stiffness matrix
[c]	Damping matrix
[H]	Frequency response matrix
[I]	Indentity matrix
[ΔM]	Mass modification matrix
[ΔC]	Damping modification matrix
[ΔK]	Stiffness modification matrix
[Φ]	Scaled modal matrix of system model
[ ]^T	Denotes the transpose matrix

We hope this booklet has answered many of your questions and will continue to serve as a handy reference. If you have other questions about structural testing techniques or instrumentation, please contact one of our local representatives or write directly to:

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